Separation graphs and their plane spanning subgraphs

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1 Introduction

Let S be a set of sistinct points of xy-plane, $\mathbf{R^2}$. Assume that no three points of S locate on a line and no two points of S have same x-coordinate or same y-coordinate. Let *conv* S be the convex hull of S and let $V(conv\ S)$ be the set of points of S on the boundary of *conv* S.

For $a,\ b \in \mathbf{R^2}$ let $B(a,b)$ denote the minimal closed box (called a *standard box*) with sides parallel to the axes, containing a and b. A pair of points $\{a,b\}$ of S is called *separated* (in S) if $A \cap B(a,b) = \{a,b\}$. Let G_S denote the *separation graph* of S, that is, the graph on the set of vertices S in which $a,b \in S$ are joined iff a,b is separated.

$$f(n) = max\{|E(G_S)| : S \subset \mathbf{R^2}, |S| = n\}.$$

A notion of separation in $\mathbf{R^d}$ was introduced by Alon, Füredi and Katchalski in [1] and they obtained that

$$f(n) \geq \lfloor n^2/4 \rfloor + n - 2 \quad \text{for all } n \geq 2.$$

This result is sharp, *i.e.* for every $n \geq 2$ there exists a set S in which the number of edges of G_S coincides with the value of the right side of the above inequation. Related problems are discussed in general dimension in [1]. In [2] Nakamigawa and Watanabe introduced a notion k-separation as a generalization of “separation” as follows. A pair of points $\{a,b\}$ of S is called k-separated if there exists a weakly monotone sequence in S with $k+1$ points containing a and b as its endpoints. Then separation means 2-separartion in this definition. For a positive integer n, let $f(n,k)$ be the smallest integer t such that every n-set $S \subset \mathbf{R^2}$ has t-separated pairs. They determined $f(n,3)$ for all n. See [2] for detail.

In this paper we study some characteristic of a separation graph. The next theorem is our main result. A term ”covered” in the theorem is slightly different from that of standared as follows: A convex hull of S, *conv* S, is *coverded* by a set $\mathcal{S}$ of some standard boxes obtained from S means that for any point $a \in conv\ S$ there exists a standared box $R \in \mathcal{S}$ and every standard box does not contain any point of S in its interior.

Theorem 1 *Let $n = |S| \geq 3$ and let $m = |V(conv\ S)|$. If every two adjacent points on the boundary of conv S such that $\{a, b\}$ is separated, then conv S is covered by at most $3n - m - 3$ standard boxes. And there exists an example needed $3n - m - 3$ standard boxes. Moreover, there exists an example whose convex hull is covered by $\lceil 3n/2 \rceil - 2$ standard boxes.*

2 Lemmas

Before describing lemmas let us recall that for any finite set S of points in $\mathbf{R^2}$ and for any non-negative integer k, if S' is obtained from S by rotating $k\pi/2$ radian or turning S over, then $G_{S'}$ is isomorphic to G_S. It is trivial but useful, so we will often use it in the discussion from now on without notice. We need two lemmas To prove Theorem 1.

Lemma 2 *Let $n = |S| \geq 3$. If every 2-subset of adjacent points on the boundary of conv S is separated, then there exists a plane spanning subgraph of G_S in which each face except the outer region is a triangle.*

Lemma 3 *Let $n = |S| \geq 3$. If every 2-subset of adjacent points on the boundary of conv S is separated and $S = V(conv\ S)$, then G_S contains a plane internal traiangulation as a subgraph.*

Here, we define some notations to prove the above lemmas.
For $a \in \mathbf{R^2}$, let $l_H(a)$ (resp. $l_V(a)$) denote the straight line passing through a and pararell to x-axis (resp. y-axis). For $a \in S$ and a vertical line l, let l^+ (resp. l^-) denote the right (resp. left) region of l. For a straight line l unless parallel to y-axis, let l^+ (resp. l^-) denote the upper (resp. lower) region of l. For two points a, b, let $l(a, b)$ denote the straight line passing through a and b. For $a \in \mathbf{R^2}$ let $x(a)$ (resp. $y(a)$) denote x-cordinate (resp. y-cordinate) of a. Let *conv* S denote the convex hull of S. Let $V(conv\ S)$ denote the set of points of S on the boundary of *conv* S. For a graph $G = G(S)$ and $a \in S$, let $N_G(a)$ denote the neighborhood of a.

Proof of Lemma 3. We apply induction on n. If $n = 3$ then the assertion holds. Now let $n \geq 4$, and assume the assertion is true for smaller sets. Let us now consider a set $S = \{a_1, a_2, \cdots, a_n\}$ satisfied with the conditions of the assertion. Without loss of generality we may assume that $x(a_1) < x(a_2) < \cdots < x(a_n)$. We divide the proof into two cases.

Case 1. All points of S are in the upper or lower region to the line $l(a_1, a_n)$.
Without loss of generality we assume that all points are in the upper region to the line $l(a_1, a_n)$ and assume that next to a_n, a_i has the largest y-coordinate point in S. By using $a_i a_n \in E(G_S)$,

we divide the boundary of *conv* S two polygons, *i.e.* the polygon with $S_1 = \{a_1, a_i, a_n\}$ as the vertex set and the other with $S_1 \setminus \{a_1\}$. Both S_1 and S_2 are satisfied with the conditions of the assertion of the lemma. By induction hypothesis both G_{S_1} and G_{S_2} have plane triangulations as subgraph, then $G_{S_1 \cup S_2} = G_S$ also contains a plane triangulation as a subgraph.

Case 2. There are points of S in both the upper regions and the lower region to the line $l(a_1, a_n)$.
Since $n \geq 4$, there exist an integer i $(1 < i < i+1 < n)$ such that $a_i a_{i+1} \in E(G_S)$ divides the boundary of *conv* S into two polygons, *i.e.* the polygon with $S_1 = \{a_1, .., a_i, a_{i+1}\}$ as the vertex set and the other with $S_2 = \{a_i, a_{i+1}, \cdots, a_n\}$. Then, In this case, by induction hypothesis $G_{S_1 \cup S_2} = G_S$ also contains a plane triangulation as a subgraph.
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Proof of Lemma 2. Let G_0 be the plane triangulation constructed in the proof in lemma 3. We apply induction on order of a set T with $V(conv\ S) \subseteq T \subseteq S$. For $T = V(conv\ S)$ the assurmption holds by lemma 3. We assume that T with $V(conv\ S) \subset T \subset S$. Choose an arbitrary point $v \in S \setminus T$. Let $\triangle pqr$ be the triangle of G_T containing v. By using lines $l_{V(r)}$ and $l_{V(r)}$, we divide $\triangle pqr$ into four regions shown as follows.

$$I_1 := \triangle pqr \cap l_V^+(r) \cap l_H^+(q)$$
$$I_2 := \triangle pqr \cap l_V^-(r) \cap l_H^+(q)$$
$$I_3 := \triangle pqr \cap l_V^-(r) \cap l_H^-(q)$$
$$I_4 := \triangle pqr \cap l_V^+(r) \cap l_H^-(q)$$

Case 1. $v \in \mathrm{I}_1$

Note that $qr \in E(G_T)$ is not the boundary of G_T. In fact, if qr is an edge on the boundary of G_T, then $qr \in E(G_S)$ must be an edge on the boundary of G_S. Then $\{q, r\}$ is not separated in S, a contradiction.
We prepare some notations to simplify the arguments below.

$$A := l_V^+(r) \cap l_V^+(v) \cap l_H^+(r)$$
$$A' := l_V^+(v) \cap l_H^+(r)$$
$$B := l_H^-(v) \cap l_H^+(q) \cap l_V^+(q)$$
$$B' := l_V^+(q) \cap l_H^+(v)$$

For any edge $e \in E_G(A, B)$ when there is no point $s \in (A' \cup B') \cap S$ with $l(vs) \cap e \neq \phi$, we define a graph $G_{T'}$ with $T' := T \cup \{v\}$ as the vertex set by:

$$E_{G_{T'}} := E_{G_T} \cup \{vp, vq, vr\}$$

If there exists such point $s \in (A' \cup B') \cap S$, we choose point s so that $|l(v,s) \cap e|$ has a minimum value among $e \in E(G_T)(A, B)$. Then $e \in E(G_T(A, B))$ if and only if $e \in E(G_T) \cap l(v, s)$.

Now let us suppose that

$$E(G_T(A, B)) =: \{a_1 b_1 = qr, a_2 b_2, \cdots, a_k b_k\}$$

where $x(a_1) < x(a_2) < \cdots < x(a_k) < \cdots < x(b_1) < \cdots < x(b_k)$, and a_i's andb_j's are not necessarily different, respectively.

Hence $y(a_1) < y(a_2) < \cdots < y(a_k) < \cdots < y(b_1) < \cdots < y(b_k)$, for otherwise, there exists an integer i such that $x(a_i) < x(a_{i+1}) < x(b_i)$ and $y(a_i) < y(a_{i+1}) < y(b_i)$, so $a_{i+1} \in R(a_i, b_i)$, which contradicts to $a_i b_i \in E(G_T)$.

Then define a graph $G_{T'}$ with $T' := T \cup \{v\}$ as the vertex set by:

$T' := T \cup \{v\}$ and

$$E(G_{T'}) := (E(G_T) \setminus E(G_T(A, B)) \cup \{vp, vq, vr\} \cup \{va_1, \cdots, va_k, vs, vb_1, \cdots, vb_k\}.$$

The graph $G_{T'}$ is a plane graph containing $v \in \triangle pqr$, so the case completes.

Each proof in the following three cases is similar to Case 1, and is omitted. We would like to note what is a set corresponding to A, A', B and B' respectively. In each case those sets are given as follows.

Case 2: $v \in \mathrm{I}_2$

$$A := l_V^-(p) \cap l_H^-(v) \cap l_H^+(r)$$
$$A' := l_H^+(v) \cap l_V^-(p)$$
$$B := l_V^+(v) \cap l_V^-(q) \cap l_H^+(q)$$
$$B' := l_V^-(v) \cap l_H^+(q)$$

Case 3: $v \in \mathrm{I}_3$

In the case there exist two couples $(A, B), (C, D)$:

$$A := l_V^-(p) \cap l_H^-(v) \cap l_H^+(p)$$
$$A' := l_V^-(p) \cap l_V^-(p) \cap l_H^+(v)$$
$$B := l_V^+(v) \cap l_V^-(r) \cap l_H^+(r)$$
$$B' := l_V^-(v) \cap l_H^+(r)$$
$$C := l_V^+(p) \cap l_V^-(v) \cap l_H^-(p)$$
$$C' := l_V^+(v) \cap l_H^-(p)$$
$$D := l_V^+(q) \cap l_H^+(v) \cap l_H^-(q)$$
$$D' := l_V^+(q) \cap l_H^-(v)$$

Case 4: $v \in \mathrm{I}_4$

$$A := l_V^+(p) \cap l_V^-(v) \cap l_H^-(p)$$
$$A' := l_V^+(v) \cap l_H^-(p)$$
$$B := l_V^+(q) \cap l_H^+(v) \cap l_H^-(q)$$
$$B' := l_V^+(q) \cap l_H^-(v)$$

■

3 Proof of Theorem 1

By lemma 2, there exists a planar subgraph G' of G in which all faces except the infinite face are triangles. Let $m = |V(conv\ S)|$ and let f be the number of faces except the infinite face. By *Euler's formula*, the total number of edges of the boundary of each face is equal to $3f + m = 2|E(G_S)|$. On the other hand, by the same formula, we have that $n - |E(G_S)| + (f + 1) = 2$. Thus $3f + m = 2|E(G_S)|$, and then $|E(G_S)| = 3n - m - 3$.

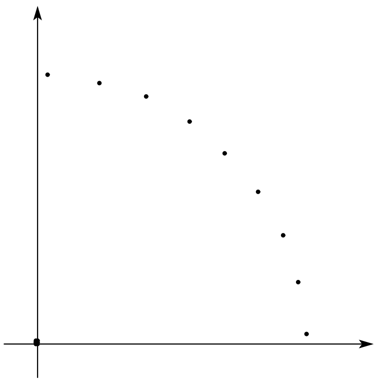

Figure 1:

Now, for any point $a \in S$ there exists a unique triangle of G', say $\triangle pqr$ containg a in the interior. It is obvious to check that $a \in R(p,q) \cup R(q,r) \cup R(r,p)$. Hence *conv* S is covered by at most $3n - m - 3$ standard boxes.

Moreover, there exists an infinite series of examples ataining to this value. Arrange the points of S ($|S| = n$, $m = |V(conv\ S)|$) such that one is in the origin and the others are on a circle in the first quadrant (see Figure 1). Then it is obvious that $3m - n - 3 = n - 3$ boxes are needed to cover *conv* S.

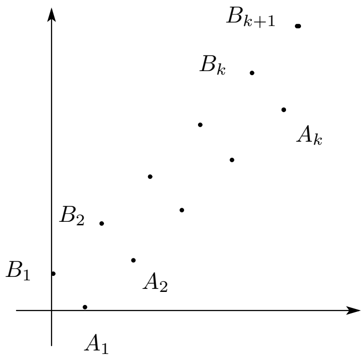

Figure 2:

For a sufficiently small real number ϵ, put

$A_t := (3t+2, 3t+\frac{1}{2}t(t+1)\epsilon)$, $B_t := (3t+\frac{1}{2}t(t+1)\epsilon, 3t+2)$.

Consider the following set (see Figure 2):

$$S := \{A_t; 0 \leq t \leq k\} \cup \{B_t; 0 \leq t \leq k\}$$

Then $|S| = 2k+2$, so $3k+1 = \frac{3}{2}n-2$ standard boxes are needed to cover *conv* S. Indeed, for each $k(0 \geq k \geq n-1)$ the quadrilateral formed by four points $A_t, B_t, A_{t+1}, B_{t+1}$ is covered by four standard boxes, $R(A_t, B_t), R(B_t, B_{t+1}), R(A_t, A_{t+1}), R(A_{t+1}, B_t)$, except four corner points. Moreover, if we consider the following set (see Figure 2):

$$S' := S \cup B_{t+1}.$$

Then $|S'| = 3k+3$, so $\lceil\frac{3}{2}n\rceil - 2$ standard boxes are needed to cover *conv* S'. ■

References

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Let S be a finite set of distinct points in the plane, $\mathbf{R}^2$. For $a, b \in \mathbf{R}^2$ let $B(a, b)$ denote the minimal closed box with sides parallel to the axes, containing a, b. A pair of points $\{a, b\}$ of S is called *separated* (in S) if $S \cap B(a, b) = \{a, b\}$. Let G_S denote the *separation graph* of S, that is, the graph on the set of vertices S in which $a, b \in S$ are joined iff a, b is separated. Notions of separation and separation graph were introduced by Alon, Füredi and Katchalski. In this paper we give the following result.
Let $n = |S| \geq 3$. If every 2-subset of adjacent points on the boundary of *conv* S is separated, then there exists a plane spanning subgraph of G_S in which each face except the outer region is a triangle. Moreover, we give an application of this result to computational geometry.