A sufficient condition for the existence of a Hamiltonian cycle in a separation graph

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1 Introduction and Definitions

Let S be a finite set of distinct points in the plane, $\mathbf{R^2}$. For $a, b \in \mathbf{R^2}$ let $B(a, b)$ denote the minimal closed rectangle with sides parallel to the axes, containing a, b. A pair of points $\{a, b\}$ of S is called *separated* (in S) if $S \cap B(a, b) = \{a, b\}$. Let G_S denote the *separation graph* of S, that is, the graph on the set of vertices S in which $a, b \in S$ are joined iff $\{a, b\}$ is separated. Let $E(G_S)$ denote the edge set of G_S and put $f(n) = max\{|E(G_S)| : S \subset \mathbf{R^2}, |S| = n\}$. The notions of separation and separation graph in $\mathbf{R^2}$ were introduced by Alon, Füredi and Katchalski in [1] and they showed that

$$f(n) = \lfloor n^2/4 \rfloor + n - 2 \quad \text{for all } n \geq 2.$$

In [2] Nakamigawa and Watanabe generalized the notion of the separation graph. In [3] Watanabe showed that if $n = |S| \geq 3$ and every pair of adjacent points on the boundary of the convex hull of S is separated, then there exists a spanning subgraph of G_S which each face except the outer region is a triangle.

In this paper, we present a sufficient condition for having a Hamilton cycle in a separation graph. We need some definitions before describing the main theorem. Let S denote a set of points of xy-plane. Throughout this paper, we assume that no three points of S lie on a line and no two points of S have the same x-coordinate or y-coordinate. Let *conv* S denote the convex hull of S and let $\mathcal{E}(conv\ S)$ denote the edge set of *conv* S. For any edge $ab \in \mathcal{E}(conv\ S)$, when $B(a, b)$ is divided by the diagonal ab to two right triangles, let Δ_{ab} denote the right triangle which contains interior points (not neccesary to be a point of S) of *conv* S. We call Δ_{ab} a *delta associated with* ab. For distinct edges $ab, cd \in \mathcal{E}(conv\ S)$, a pair $\{ab, cd\}$ is said to be *closed* if $\Delta_{ab} \cap \Delta_{cd}$ forms a polygon (see Fig.1). Throughout this paper whenever werite $ab \in \mathcal{E}(conv\ S)$, w.l.o.g the points a,b lie on the boundary of $convS$ in clockwise order.

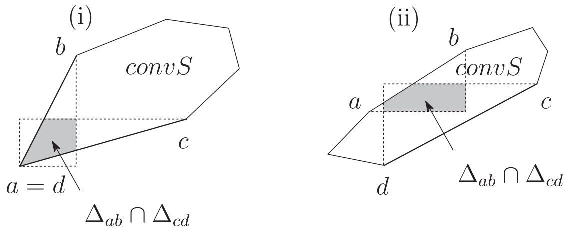

Figure 1: Examples of closed pair

If there exists a closed pair of edges in $\mathcal{E}(conv\ S)$, G_S is not always Hamiltonian. In Fig.1 (i), if $\Delta_{ab} \cap \Delta_{cd}$ contains only one point d of S in its interior, then G_S is non-Hamiltonian, since the degree of a in G_S is one. A pair of deltas $\{\Delta_{ab}, \Delta_{cd}\}$ is said to be *good* if it satiffies the following conditions:
(i) a, b, c, d lie in the boundary of *conv* S in clockwise order;
(ii) $\Delta_{ab} \cap \Delta_{cd} = \emptyset$;
(iii) the extending line of a horizontal or vertical edge of Δ_{ab} hits Δ_{cd} and simultaneously the extending line of a horizontal or vertical edge of Δ_{cd} hits Δ_{ab}.

If $\{\Delta_{ab},\ \Delta_{cd}\}$ is a good pair and the slope of ab has a positive sign, by the previous assumption, then we may assume that $x(a) < x(d) < x(b) < x(c)$ and $y(d) < y(c) < y(a) < y(b)$. And we may assume that aa' and ba' are the legs of Δ_{ab} and cc', and dc' are the legs of Δ_{cd}. Then the extending line of ba' hits at a point, say b', in the leg cc' of Δ_{cd} and the extending line of dc' hits at a point, say d', in aa'. The rectangle $a'b'c'd'$ is said to be *associated with* $\{\Delta_{ab}, \Delta_{cd}\}$ (see Fig. 2).

If there exists a pair $\{ab, cd\}$ of edges in $\mathcal{E}(conv\ S)$ such that an associated rectangle with $\{\Delta_{ab}, \Delta_{cd}\}$ contains a point of S, G_S is not always Hamiltonian. The set $S = \{a, b, c, d, e, f, g, h, z\}$ shown in Fig. 3 is such an example. In this case, the associated rectangle with $\{\Delta_{ab}, \Delta_{cd}\}$ contains z, and the associated rectangle with $\{\Delta_{ef}, \Delta_{gh}\}$ also contains z.

The following is our main theorem.

Theorem 1 *Assume that* $|\mathcal{E}(conv\ S)| \geq 4$ *and* $\mathcal{E}(conv\ S)$ *contains no closed pair. If there exists an associated rectangle with a good pair of deltas which contains no points of* S*, then* G_S *is Hamiltonian.*

In section 2 we shall prepare some lemmas to prove the theorem and in section 3 we shall prove the theorem.

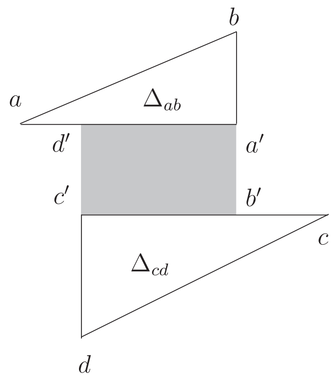

Figure 2: The rectangle $a'b'c'd'$ associated with $\{\Delta_{ab}, \Delta_{cd}\}$

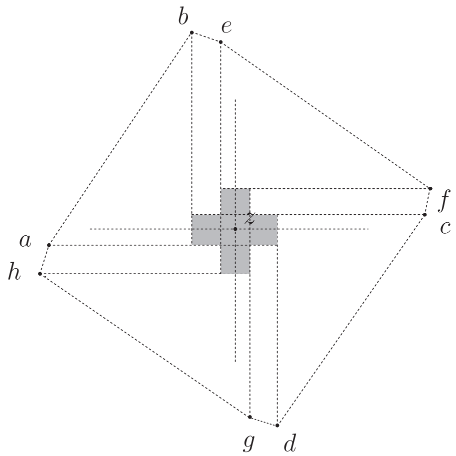

Figure 3: The graph G_S with $S = \{a, b, c, d, e, f, g, h, z\}$ is non-Hamiltonian

2 Lemmas

We begin by observing the following note. For any finite set S of points in $\mathbf{R^2}$ and for any non-negative integer k, if S' is obtained from S by rotating $k\pi/2$ radian or turning S over, then $G_{S'}$ is isomorphic to G_S. It is trivial but we will often use it in the discussion from now on.

Lemma 2 *Let $|\mathcal{E}(conv\ S)| \geq 4$. If $\mathcal{E}(conv\ S)$ contains no closed pair, then there exists a good pair of deltas.*

Proof. Let z denote the lowest point of S. Then there exist adjacent points a, b with z in the boundary of *conv S* such that $x(a) < x(z) < x(b)$, since $\mathcal{E}(conv\ S)$ has no closed pair. If the line segment $\Delta_{za} \cap \Delta_{bz}$ is extended to the vertical direction, then it hits some delta Δ_{cd}. We may assume w.l.o.g that cd has the same slope to $az \in \mathcal{E}(conv\ S)$. If the vertical edge of Δ_{cd} is extended to the vertical direction, then it hits some delta Δ_{ef}, since $\mathcal{E}(conv\ S)$ has no closed pair again. Since the sign of the slope of cd and the slope of ef have the same sign, $x(f) < x(c) < x(e) \leq x(z)$. Hence $\{cd, ef\}$ is a good pair as required. □

An *orthogonal* polygon is a polygon with edges which alternate between horizontal (zeroslope) and vertical (infinite slope). Let $A_1, A_2, \cdots, A_m$ be a consecutive sequence of the vertices of a polygon is called *monotone with respect with to a line L* if the projections of $A_1, A_2, \cdots, A_m$ onto L are ordered the same as in the sequence $A_1, A_2, \cdots, A_m$, that is, there is no doubling back in the projection as the sequence is traversed.
Let P_S denote the boundary of the region obtained by removing all the deltas from *conv S*.

Lemma 3 *Let $|\mathcal{E}(conv\ S)| \geq 4$. Assume that $\mathcal{E}(conv\ S)$ has no closed pair. Then P_S is an one-connected orthogonal polygon and monotone with respect to both vertical line and horizontal line.*

Proof. From the definition, P_S is composed by edges or parts of legs of some deltas, so P_S is clearly an orthogonal polygon. Since $\mathcal{E}(conv\ S)$ has no closed pair, for any two edges $ab, cd \in \mathcal{E}(conv\ S)$, $\Delta_{ab} \cap \Delta_{cd}$ does not form a polygon, so P_S is still one-connected. On the other hand, any vertical or horizontal straight line which intersects the boundary of *conv S* at two points of *conv S* intersects at most two deltas. It implies P_S is monotone. □

Lemma 4 *Assume that $p, q, r, s \in S$ form a trapezoid with base ps such that $pq \in \mathcal{E}(conv\ S)$ and ps, qr are parallel to x-ax. Then there exists a (p, q)-path (*i.e. *between p and q) in G_S which has p, q as endpoints and connects all points of S in the interior of the trapezoid $pqrs$.*

Proof. By the assumption, we may assume w.l.o.g. that $pq \in \mathcal{E}(conv\ S)$, and ps, qr are parallel to x-ax. If the ordered list with respect to y-coordinate of all the points of S in the interior of the trapezoid $pqrs$ is $a_1(=p), a_2, \cdots, a_m(=q)$, then each $\{a_i, a_{i+1}\}$ $(1 \leq i \leq m-1)$ is separated, so $a_i a_{i+1} \in E(G_S)$ for all $1 \leq i \leq m-1$. Hence $a_1 a_2 \cdots a_m$ is a path in G_S as required. □

Corollary 5 *If three points $p, q, s \in S$ form a right triangle such that the legs are ps, qs and ps is parallel to x-ax, then there exists a (p, q)-path in G_S which connects all points of S in the interior of the triangle pqs.*

3 Proof of Main Theorem

If $|\mathcal{E}(conv\ S)| = 4$ and $\mathcal{E}(conv\ S) = \{a_1, a_2, a_3, a_4\}$, by the assumption, any pair of non-adjacent two edges of $\mathcal{E}(conv\ S)$ forms a good pair, then there exist no points of S in the interior of $convS$. By Cor. 5, for each i there exists (a_i, a_{i+1})-path in G_S. Therefore by combining these four pathes we obtain a Hamilton cycle of G_S.

Assume that $|\mathcal{E}(conv\ S)| \geq 5$. By Lemma 3, P_S is an one-connected and monotone orthogonal polygon with respect to both vertical line and horizontal line. By Lemma 2, there exists two deltas Δ_{ab}, Δ_{cd} such that the interior of its associated rectangle (say $a'b'c'd'$) contains no points of S.

We assume w.l.o.g that

$$x(a) < x(d) < x(b) < x(c) \text{ and } y(a) < y(b).$$

Then $y(d) < y(c) < y(a)$.

The edges $ab, cd \in \mathcal{E}(conv\ S)$ divide the boundary of $conv\ S$ into two brokenlines. We denote by B_1 the broken line having a, d as endpoints and similarly denote by B_2 the the broken line having b, c as endpoints. Let A, B denote the point whose x-coordinate is minimum or maximum among all points of S, respectively. Similarly let C,D denote the point whose whose y-coordinate is minimum or maximum among all points of S, respectively. Then A, C lie in B_1 and B, D lie in B_2. By removing the rectangle $a'b'c'd'$ from P_S, we have at most two orthogonal polygons P_1, P_2. It is possible that P_1 or P_2 is empty. In deed if $|\mathcal{E}(conv\ S)| = 5$ then bothe P_1 or P_2 is empty. We may assume that P_1 lies in P_S facing to the vertical edge of Δ_{ab}. Then P_2 lies in G_S facing the vertical edge of Δ_{cd}. We can divide each P_i into a collection of more small rectangles by two steps as follows.

First step: In P_1, extend to downward the vertical leg of the delta of each edge of B_1 until it hits some delta. On the contrary in P_2, extend to upward the vertical leg of the delta of each edge of B_2 until it hits some delta.Then the interior of each P_i is divided by

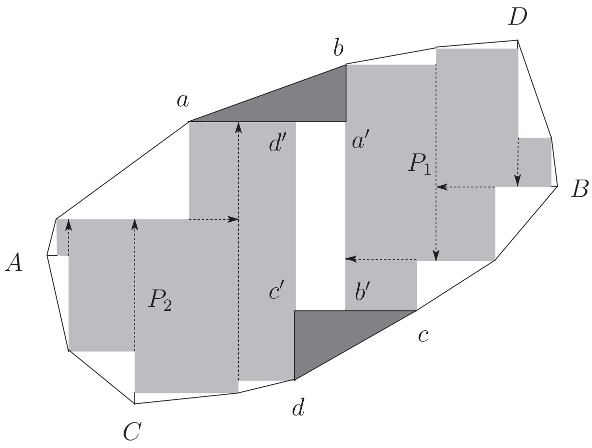

Figure 4: Rectangles of $\mathcal{R}$ in the neighbor of A

some rectangles and some orthogonal rectangles containing some verticess whose interior angles are reflex.

Second step: In each subdivision obtained by the first step, the side opposite to a reflex vertex is an edge of $\mathcal{E}(conv\ S)$. Then for each subdivision T containing reflex vertices and for every reflex vertex t of T, extend the horizontal edge of the delta containing t until it hits some edge of T.

Let $\mathcal{R}$ denote the collection of the rectangle $a'b'c'd'$ and all the rectangles obtained by the above steps. Thus for each rectangle $R \in \mathcal{R}$, there exists exactly one delta, say Δ_{xy}, which either the vertical leg or the horizontal leg of Δ_{xy} coinsides with some side of R. Indeed, for any orthogonal polygon with reflexes in the subdivision in step 1 its all reflexes vanish in step 2.

Consider a couple of R and Δ_{xy} and absorbe R and Δ_{xy} to get a trapezuim. All rectangles of $\mathcal{R}$ except rectangle $a'b'c'd'$ can contribute to make this coupling (see Fig.4). And all deltas except two deltas (either of two deltas having A as vertices and either of two deltas having B as vertices) contribute this pairing (see Fig.5). Then we obtain a collection $\mathcal{T}$ of the trapeziums which gives a new partition of $P_1 \cup P_2$. Hence for each edge $xy \in \mathcal{E}(conv\ S)$ which contributes to the above absorbing there exists a (x, y)-path in G_S passing through all points of S in a trapezium of $\mathcal{T}$ which includes Δ_{xy}. Let $\mathcal{H}$ be the set of all paths described above. On the other hand, it is trivial that for each delta Δ_{xy} (there exist at most two such deltas) which does not contribute the abobe coupling there exists a (x, y)-path in G_S passing through all points of S in Δ_{xy}. From the assumption, the interior of rectangle $a'b'c'd'$ contains no points of S. Hence each point of S is in some path of $\mathcal{H}$.

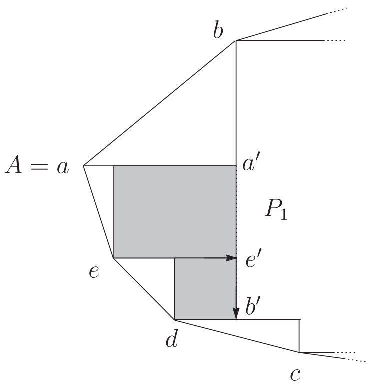

Figure 5: Orthogonal polygons P_1, P_2 in the interior of P_S

Since $|\mathcal{E}(conv\ S)| \geq 4$, any two paths of $\mathcal{H}$ contains at most one point of S in common. Thus $\mathcal{H}$ consists a Hamilton cycle of G_S. $\square$

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A Sufficient Condition for the Existence of a Hamilton Cycle in a Separation Graph

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Let S be a finite set of distinct points in the plane, $\mathbf{R^2}$. For $a, b \in \mathbf{R^2}$ let $B(a, b)$ denote the minimal closed box with sides parallel to the axes, containing a, b. Let G_S denote the *separation graph* of S, that is, the graph on the set of vertices S in which $a, b \in S$ are joined iff $S \cap B(a, b) = \{a, b\}$. The notions of separation and separation graph in $\mathbf{R^2}$ were introduced by Alon, Füredi and Katchalski in 1985. In this paper, we present a sufficient condition for the existence of a Hamilton cycle in a separation graph.