Minimum sets in an A_2-lattice whose component does not induce any equilateral triangle

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1 Introduction

In this paper we will study a certain problem in equilateral triangle lattices. In [1] the problem (Problem 25) is described as:

> Ten coins are arranged as shown in Figure 1-A. What is the minimum number of coins we must remove so that no three of the remaining coins lie on the vertices of an equilateral triangle?

If we remove coins as shown in Figure 1-A, points x,y and z still consist of an equilateral triangle. On the other hand if we remove coins as shown in Figure 1-B, all the lattice points except the remaining four coins have an equilateral triangle. In this paper we consider a generalized problem of the above problem. We start by giving some definitions.

Consider a equilateral triangle $T_n = ABC$ each of whose segment has length n. We mark each point p on each peripheral segment S which has an integer distance from the endpoints of S, and add all straight segments passing through these points to be parallel to peripheral segments, as shown in Figure 2. The number n is called the *size* of T_n.

Let $\mathcal{T}_n$ be the set of triangle in T_n. A subset H of $V(T_n)$ be a *destroyer* if

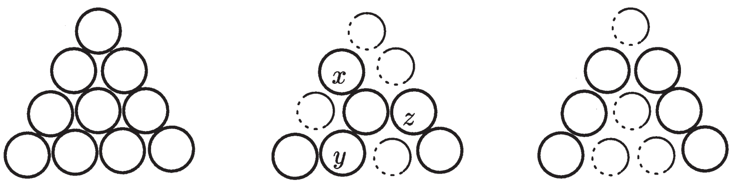


Figure 1-A Figure 1-B Figure 1-C

Figure 1: A coin problem

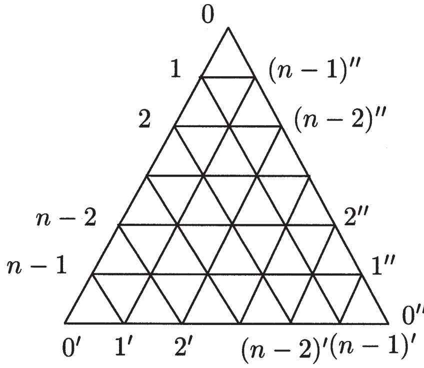


Figure 2: T_n

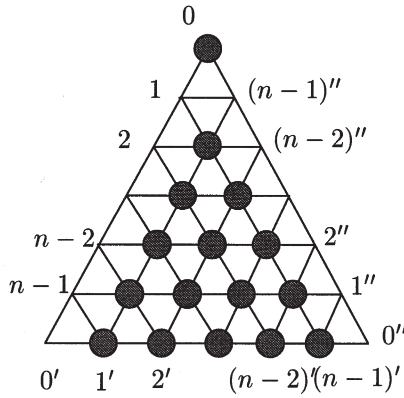


Figure 3: B_n

every three points of $V(T_n) - H$ consist no triangle. Since $\{0, 0', 0''\}$ consists of a triangle for any n, we suppose w.l.o.g. that 0 is a vertex belong to any destroyer. In [2] Nakamoto and Watanabe showed that the number $f(n)$ of triangles in T_n is given by the following:

$$f(n) = \lfloor \frac{n(n+2)(2n+1)}{8} \rfloor.$$

Let $\mathcal{D}_n$ be the set of minimum destroyers. We denote by $\xi(n)$ the size of any set of $\mathcal{D}_n$. Let be B_n is the configuration as shown in Figure 3. We note that B_n is a destroyer of T_n. Therefore we get

$$\xi(n) \leq 1 + \frac{n(n-1)}{2}.$$

Our result is the following.

Theorem 1 $\mathcal{D}_n = \{B_n\}$ *for* $i \in \{1, 2, 3, 4\}$, *and* $\mathcal{D}_5 = \{B_5, N_1, N_2\}$

where $\{N_1, N_2\}$ *are the configurations as shown in Figure 7.*
Moreover, $\xi(i) = 1 + \dfrac{n(n-1)}{2}$ *for each* $n \in \{1,2,3,4,5\}$.

The proofs are given in the next section for T_1, T_2, T_3, T_4 and in the section 3 for T_5.

2 The cases T_1, T_2, T_3, T_4

In this section we will prove Theorem 1 in the cases of T_1, T_2, T_3, T_4, and T_5 in the next section . From now on, for the sake of convenience we often denote a set by the product of the elements , for example $014' = \{0, 1, 4'\}$.

We first consider T_1. Since T_1 has a single triangle with three vertices o, o', o'', it is clear that $\mathcal{D}_1 = \{B_1\}, \xi(1) = |B_1|1$. For T_2, B_2 is a destroyer, and every three vertices of $V(T_2) \setminus V(B_2)$ consists no triangle. Since $00'0''$ and $11'1''$ are disjoint triangles, we get $\mathcal{D}_2 = \{B_2\}$, and $\xi(2) = |B_2| = 2$.
Next we consider T_3. Call 3 the vertex which does not lie on the boundary of T_3. Let H be any destroyer of T_3. Since $132'', 20'1', 1''2'0'' \in \mathcal{T}_n$, then $|H| \geq 4$. On the other hand $B_3 \in \mathcal{D}_3$, then $|H| \leq 4$. Thus $|H| = 4$ and $\xi(3) = 4$. We will show that $\mathcal{D}_3 = \{B_3\}$. To contradict we assume this is fault. From Lemma 2 we get that $|S_i \cap H| \leq 1$, then $1, 2, 0', 0'', 1'', 2'' \notin H$. Since $123, 31''2'' \in \mathcal{T}$, $3 \in H$. Similarly, $1' \in H$ since $20'1', 1'0''2'' \in \mathcal{T}$, and $2' \in H$ since $10'2', 1'0''2'' \in \mathcal{T}$. Thus $H \equiv B_3$, which is a contradiction. Thus we obtain $\mathcal{D}_3 = \{B_3\}$, and $\xi(3) = |B_3| = 4$.
Here we need the following Lemmas to discuss about T_4.

Lemma 2 *If* $n \leq 2$ *and* $\mathcal{D}_{n-1} = \{B_{n-1}\}$, *then any* $H \in \mathcal{D}_n$ *satisfies*
$$|S_i \cap H| \leq n-1 \quad \textit{for } i \in \{1,2,3\}.$$
Moreover, if there exists $i \in \{1,2,3\}$ *such that* $|S_i \cap H| = n-1$, *then* $H - S_i \equiv B_{n-1}$.

Lemma 3 *Suppose that* $2 \leq n \leq 5$. *Then for any* $H \in \mathcal{D}_n$ *with* $H \not\equiv B_n$,
$$|S_i \cap H| \leq n-2 \quad \textit{for } i \in \{1,2,3\}.$$

Proof of Lemma 3: We assume that there exists $i \in \{1,2,3\}$ such that $S_1 \cap H \geq n$. Then we get $1 + \frac{1}{2}n(n-1) = |B_n| \geq |H| = |S_1 \cap H| + |H - S_i| \geq n + |H - S_i|$ Since $H - S_i$ is a destroyer of T_{n-1}, $\frac{1}{2}(n^2 - 3n + 2) \geq |H - S_i| \geq |B_{n-1}| = 1 + \frac{1}{2}n(n-1) = 1 + \frac{1}{2}(n^2 - 3n + 4)$, which is a contradiction. Assume that $S_1 \cap H \leq n-1$. Then, $|H - S_i| = |H| - |S_i \cap H| = |H| - n + 1 \leq |B_n| - n + 1 = \frac{1}{2}(n^2 - 3n + 4) = |B_{n-1}|$. By the assumption, $\mathcal{D}_{n-1} = \{B_{n-1}\}$, hence $H - S_i \equiv B_{n-1}$. (The end of the proof of Lemma 3)

Proof of Lemma 4: For any $\mathcal{D}_n$ with $H \equiv B_n$, from Lemma 3 we get $|S_1 \cap H| \leq n-1$. If $|S_1| \cap H| = n-1|$, from Lemma 3 we get that $H - S_1 \equiv B_{n-1}$. Since $T_n - S_1 \equiv T_{n-1}$, $|S_1| \cap H| \geq 1 + (n-1) = n$, a contradiction. Thus $|S_1| \cap H| \leq n-1$. We assume that $|S_2| \cap H| \leq n-1$. Then by repeating the arguments we get $|H - S_2| \equiv B_{n-1}$. If $0', 0'' \in H$ then $S_2 \cap H = \{1', 2', \cdots (n-1)'\}$, and $H \equiv B_n$, a contradiction. Therefore there exists $i' \in \{1', 2', \cdots, (n-1)'\}$ such that $i' \notin H$. On the other hand $ii'i'' \in \mathcal{T}$, so $i' \in H$, a contradiction. Hence we get $|S_2 \cap H| = n-2$. (The end of the proof of Lemma 4)

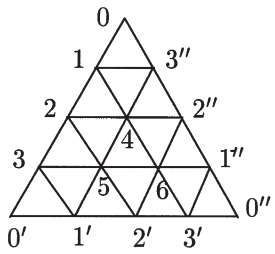


Figure 4: T_4

Now we return to prove Theorem 1. We will show that $\mathcal{D}_4 = \{B_4\}$. Choose $H \in \mathcal{D}_4$ such that $H \not\equiv B_4$. By Lemma 4, we get $|S_i \cap H| \leq 2 \ \ (i = 1, 2, 3)$.

Claim 1. $0' \notin H$. Moreover, $0'' \notin H$ by the symmetry.
proof of Claim 1: Assume that $0' \in H$. Then $|(S_1 \cup S_2 \cup S_3) \cap H| \geq 3$. By Lemma3, $|S_i \cap H| \leq 2 \ \ (i = 1, 2, 3)$. Hence, $|(S_1 \cup S_2 \cup S_3) \cap H| = 3$. However, $ii'i'' \in \mathcal{T}$ for any $i \in \{1, 2, 3\}$. Thus $|(S_1 \cup S_2 \cup S_3) \cap H| \geq 6$, which is a contradiction.
Since $0''i'(4-i)'' \in \mathcal{T}$ for any $i \in \{1, 2, 3\}$, $i' \in H$ or $(4-i)'' \in H$. Then $|S_2 \cap H| \geq 3$ or $|S_3 \cap H| \geq 3$, a contradiction. Thus we get $0' \notin H$.

Claim 2. $|S_2 \cap H| \geq 2$.
proof of Claim 2: Since $0'i'(4-i) \in \mathcal{T}$ for any $i \in \{1, 2, 3\}$, then $i \in H$ or $4-i \in H$. If $|S_2 \cap H| \leq 1$, then $|S_1 \cap H| \geq 3$, a contradiction.

From Claim 2 and Lemma 4 we get $|S_2 \cap H| = 2$. Then we get $S_2 \cap H = \{1', 2'\}$ or $S_2 \cap H = \{1', 3'\}$ by the symmetry. If $S_2 \cap H = \{1', 2'\}$ then $1 \in H$, since $1, 0'3' \in \mathcal{T}$. By Lemma 4, we obtain $2, 3 \notin H$. Since $33'3'' \in \mathcal{T}$ we get $3'' \in H$. By Lemma 4, we obtain $2'', 1'' \notin H$. Since $3'0''1'' \in \mathcal{T}$. Hence $1'' \in H$, a contradiction. In the case $S_2 \cap H = \{1', 2'\}$, we can also get a contradiction. Thus we obtain $\mathcal{D}_4 = \{B_4\}$, and $\xi(4) = 7$.

3 The case T_5

Now we discuss T_5. We give the vertices of T_n labels as shown Figure 5. For any $H \in \mathcal{D}_5$, $|H| \leq 11$ since $B_5 \in \mathcal{D}_5$. From now on we use the following notations:

$$S_1 := \{0, 1, 2, 3, 4, 0'\}, \ \tilde{S_1} := S_1 \cup \{4'', x, x', z', 1'\},$$
$$S_2 := \{0', 1', 2', 3', 4', 0''\}, \ \tilde{S_2} := S_2 \cup \{4, z, y', z, 1''\},$$
$$S_3 := \{0'', 1'', 2'', 3'', 4'', 0\}, \ \tilde{S_3} := S_3 \cup \{4', z, y, x, 1\}.$$

From now on we assume that $H \equiv B_5$.

Claim 3. $|S_1 \cap H| \leq 3$. Moreover, $|S_3 \cap H| \leq 3$ by the symmetry.
proof of Claim 3: Since $\xi(4) = 7$, we get $|S_1 \cap H| \leq 4$. Assume that $|S_1 \cap H| = 4$. Note that $H - H \cap S_1 = H - S_1$ is a destroyer of T_4. $H - H \cap S_1$ is either of the three destroyers shown as Figure 6. In the case Figure 6a, we note that $1', 2', 3', 4', 1'', 2'', 3'', 4'' \notin H$. Since $ii'i'' \in$T or $i \in \{1, 2, 3, 4\}$, $i \in H$, then we get $|S_1 \cap H| \geq 5$, a contradiction. In this case Figure 6b, Since $11'1'' \in \mathcal{T}$, then

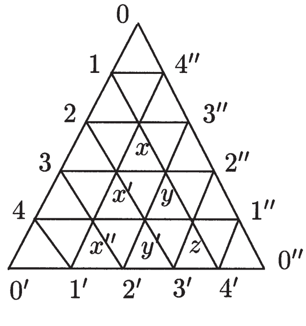


Figure 5: T_5

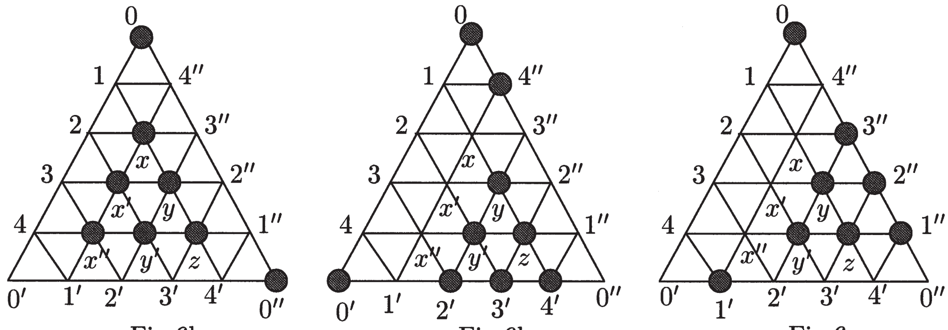


Figure 6: T_5

$1 \in H$. Similarly, we get $2,3,4 \in H$. Then $|S_1 \cap H| \geq 5$, which is a contradiction. In this case Figure 6c, if $4'', x, x', z' \notin H$, then $1,2,3 \in H$. Since $44'4'' \in \mathcal{T}$. Hence $|S_1 \cap H \geq 5|$, a contradiction. (The end of the proof of Claim 3)

Claim 4. $0' \notin H$. Moreover, $0'' \notin H$.
proof of Claim 3: Assume that $0' \in H$. From Claim 1, $|S_i \in H| \leq 3$ for $i = 1,3$. If $0'' \in H$ then $|(S_1 \cup S_2 \cup S_3) \cap H| \leq 6$, since $|S_i \cap H| \leq 3$. On the other hand, $ii'i'' \in \mathcal{T}$ for $i \in \{1,2,3,4\}$, and we can not destroy all of the four independent triangles. Then $0'' \notin H$.
Since $0''i'(5-i)'' \in \mathcal{T}$, we get

Fact 5. $i' \in H$ or $(5-i')'' \in H$, $|S_2 \cap H| = |S_3 \cap H| = 3$.

Subclaim 4.1. $S_2 \cap H = \{0', 1', 4'\}$ or $S_2 \cap H = \{0', 2', 3'\}$.
proof of Claim 4.1:It is sufficient to show that $1' \in H$ iff $4' \in H$. By symmetry, if $4' \in H$ implies $1' \in H$, then $1' \in H$ implies $4' \in H$. Assume that $1' \in H$. By Fact 5 we get $4'' \in H$. If $4 \notin H$ then $4' \in H$. Hence if $4 \in H$ then $1,2,3 \notin H$. Notice that $2' \notin H$. Indeed, if $2' \in H$ then $3'' \notin H$, so $3' \in H$. Hence $|S_2 \cap H| \geq 4$, a contradiction. Similarly we get $3' \notin H$. By Fact 5 we get $4' \in H$. (The end of the

proof of Subclaim 4-1).
From Subclaim 4.1 and the symmetry, we may assume w.l.o.g. that $S_2 \in H = \{0', 1', 4'\}$ or $S_3 \in H = \{0', 2'', 3''\}$. Then $2', 3', 1'', 4'' \notin H$ by Fact 5 . Since $2'3'y' \in \mathcal{T}$, we get $y' \in H$. Similarly we get $y', z', y, x \in H$. Then the number of the other vertices is at most one. On the other hand $23x', 14z \in\in \mathcal{T}$ are independent, a contradiction. (The end of the proof of Claim 4)

Claim 5. $|S_2 \cap H| \geq 2$.
proof of Claim 5: If $i' \notin H$ then $(5-i) \in H$, since $0'i'(5-i) \in \mathcal{T}$. Therefore, if $|S_2 \in H| \leq 1$ then $S_1 \in H \geq 4$, which contradict to Claim 4. (The end of the proof of Claim 5)

Claim 6. $|S_2 \cap H| \geq 3$.
proof of Claim 6: From Claim 5, we get $|S_2 \cap H| \geq 2$. Assume that $|S_2 \cap H| = 2$. By the symmetry, we get $S_2 \cap H = \{1', 2'\}$, $\{1', 3'\}$, $\{1', 4'\}$ or $\{2', 3'\}$. If $S_2 \cap H = \{1', 2'\}$ then $33'3'' \in \mathcal{T}$, which contradict to Claim 3. Simiraly, if $S_2 \cap H = \{1', 3'\}$ then $32'2'' \in \mathcal{T}$, a contradiction. If $S_2 \cap H = \{1', 4'\}$ then $x \in H$ since $14''x \in \mathcal{T}$. Similarly, we get $y', z, z' \in \mathcal{T}$. However $1''3'y \in \mathcal{T}$, which is a contradiction. If $S_2 \cap H = \{2', 3'\}$ then $x' \in H$ since $23x' \in \mathcal{T}$. Similarly, we get $y', z, z' \in \mathcal{T}$. $y, x, y' \in \mathcal{T}$. However $21'z \in \mathcal{T}$, which is a contradiction. (The end of the proof of Claim 6)

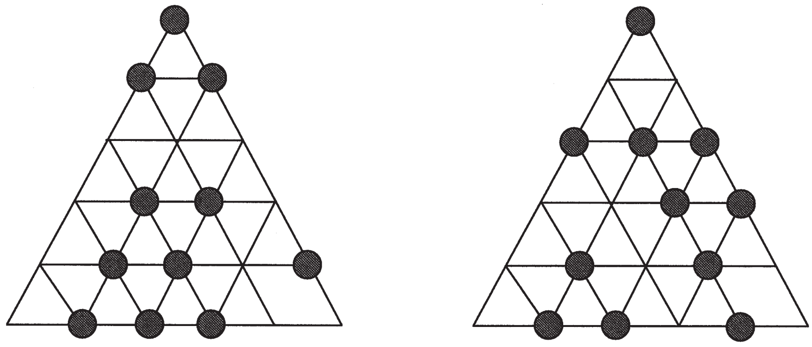
From Claim 6. we obtain $|S_2 \cap H| \geq 3$ or 4.

Case 1. $S_1 \in H = 4$.
Then $|H - S_2| = |H| - |S_2 \in H| \leq |B_5| - 4 = 7 = |B_4|$. Then $H - S_2 \equiv B_4$, that is $H \equiv B_5$.

Case 2. $|S_1 \in H| = 3$.
By the symmetry, w.l.o.g, we may assume that $|S_2 \cap H| = \{1', 2', 3'\}$ or $|S_2 \cap H| = \{1', 2', 3'\}$ *Subase 2-1.* $|S_2 \cap H| = \{1', 2', 3'\}$
Since $0'4'1, 4'0''1'' \in \mathcal{T}$, we get $1, 1'' \in H$.

Claim 7. If $4' \in H$, then $4'' \in H$.
proof of Claim 7: If $4'' \notin H$, then $4 \in H$. Assume that $4' \notin H$. Since $44'4'' \in \mathcal{T}$, $4 \in H$. Then $2, 3 \notin H$, since $|S_1 \cap H| \leq 3$. Hence $x', y, z \in H$, since $23x', 24''y, 34''z \in \mathcal{T}$. Then the number of the other vertices to allow as vertices of H is at most one. However $2''xy, 3''z'4' \in \mathcal{T}$ are independent, a contradiction. (The end of the proof of Claim 7)
From Claim 7, we get $4'' \in H$. Then $2'', 3'' \notin H$. Therefore $y, y', z \in H$, since $2''''3''y, 2''4'y', 3''4'z' \in \mathcal{T}$. Then the number of the other vertices to allow as vertices of H is at most one. However, $23x'$, $3''zx' \in \mathcal{T}$. Thus $x' \in H$. Thus we get a destroyer as shown in Figure 7. We call the configuration $N_2 - conf_1$. *Case 2-2.* $|S_2 \cap H| = \{1', 2', 4'\}$
Then $2, 2'' \in H$, since $0'3'2, 0''3'2'' \in \mathcal{T}$.

Claim 8. $3 \notin H$.
proof of Claim 8: Assume that $3 \in H$. Then $1, 4 \in H$, since $|S_1 \cap H| \leq 3$. $|S_3 \in H| = 3$. Indeed, if $1'', 3'', 4'' \notin H$ then $z', y' \in H$, since $1''4''z', 1''3''y' \in \mathcal{T}$.



N_1 – conf. N_2 – conf.

Figure 7: N_1 – configuration. and N_2 – configuration.

However, $1x'3'' \in H$, which is a contradiction. If $1'' \in H$, then we get $x' \in H$, since $13''x' \in \mathcal{T}$. Then $|H| = 11$, but $3'z'y \in H$, which is a contradiction. Then $1'' \notin H$. If $3'' \in H$, we get $z' \in H$, since $1''4''z7 \in \mathcal{T}$. Then $|H| = 11$, but $x'yy' \in H$, a contradiction. If $4'' \in H$, we get $y' \in H$, since $1''3''y' \in \mathcal{T}$. Then $|H| = 11$, but $3'yz' \in H$, a contradiction. (The end of the proof of Claim 8)

From Claim 8, if $3 \notin H$ then $3'' \in H$, since $33'3'' \in \mathcal{T}$. Then $1'', 4'' \notin H$, since $|S_3 \cap H| \leq 3$. Since $1''3'y \in \mathcal{T}$, we get $y \in H$. Similarly we get $z', z, x \in H$, since $4''1''y, 34''z, 14''x, 3y'x \in \mathcal{T}$. Thus we get a destroyer as shown in Figure 8. We call the configuration $N_2 - conf_2$.
From the above we checked on all cases, and it turned out that three configurations $B - 5.$, $N_1 - \text{conf}_2$. and $N_2 - \text{conf}_2$. are all the minimal destroyers. We clearly get $\xi(5) = 11$. This completes the proof of Theorem 1.

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Minimum sets in an A_2-lattice whose component does not induce any equilateral triangle

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Consider an equilateral triangle $T_n = ABC$ each of whose segment has length n. We mark each point p on each peripheral segments S which has an integer distance from the endpoints of S, and add all straight segments passing through these points to be parallel to peripheral segments. A subset H of $V(T_n)$ be a ***destroyer*** if every three points of $V(T_n) - H$ consist no triangle. Let $\mathcal{D}_n$ be the set of minimum destroyers. We denote by $\xi(n)$ the size of any set of $\mathcal{D}_n$. We will show that $\xi(n) = 1 + \dfrac{n(n-1)}{2}$ for each $n \in \{1, 2, 3, 4, 5\}$, and determine all the configurations attaining to $\xi(n)$