Quantum Field Theory of Dual Fock Space

Masayoshi MIZOUCHI
Faculty of Life Science
Kurashiki University of Science and the Arts,
2640 Nishinoura, Tsurajima-cho, Kurashiki-shi, Okayama 712-8505, Japan
(Received October 4, 2006)

1. Introduction

This paper presents a method for solving a basic equation of quantum field theory. We deal only with the second order diagrams of quantum electrodynamics (QED) as the simplest and the well-known example.

This paper is organized as follows. In section 2 the definition of *the frequency part exchange*[1] is given. In section 3 we construct quantum field theory with dual Fock space. In sections 4 and 5 it is presented how to calculate electron self-energy by using the method obtained in sections 2 and 3. In the final section a few significant aspects related closely to our present task are pointed out.

2. Definition of frequency part exchange

Every field operator appearing in a theory is interchanged into its opposite frequency part. For example, in the case of QED, it means that

$$\psi^{(\pm)} \longrightarrow \psi^{(\mp)}, \quad \overline{\psi}^{(\pm)} \longrightarrow \overline{\psi}^{(\mp)}, \quad A_\mu^{(\pm)} \longrightarrow A_\mu^{(\mp)}.$$

The normal order product is transformed into *reversed* normal order product, that is,

$$N(\phi_1\phi_2\cdots\phi_l) \longrightarrow N'(\phi_1\phi_2\cdots\phi_l).$$

Where, each field operator ϕ_j must includes the both frequency parts:

$$\phi_j = \phi_j^{(+)} + \phi_j^{(-)}.$$

Every propagator is transformed as follows.

$$\underline{\phi_i\phi_j} = T(\phi_i\phi_j) - N(\phi_i\phi_j) \longrightarrow \overline{\phi_i\phi_j} = T(\phi_i\phi_j) - N'(\phi_i\phi_j).$$

Where the term $T(\cdots)$ is indeed invariant under the frequency part exchange. Moreover, Fock space $\mid t >$ is also transformed into *counter Fock space* $\mid t >'$, that is

$$\mid t >= U(t,t_0)\mid t_0 > \longrightarrow \mid t >' = U'(t,t_0)\mid t_0 >',$$

where *the solution* $\mid t >'$ is corresponds actualy to *the negative energy state in quantum field theory.* We can not avoid this kind of solution. This solution is inevitable because of the symmetric property of the basic equation. This property is intrinsic for this basic equation. If we perform the above replacement for the ordinary Wick's theorem we obtain *the negative energy quantum field theory* with the negatively divergent self-energy. The method of how to obtain $U'(t,t_0)$ is explained in the next section.

3. The modified solutions

Taking account of difinitions given in previous section, we are able to solve the equation

$$i\frac{\partial}{\partial t}(U(t,t_0)+U'(t,t_0))=H_I(t)(U(t,t_0)+U'(t,t_0)),$$

under the conditions of $U(t_0,t_0)=I$, $U'(t_0,t_0)=I$. However, if we assume $U'\neq U$, we can not apply the iteration method to obtain U and U' separately. We convert this equation to integral equation. However U and U' are not separable each other at this stage. This separation has misled us for a long time! The only solution we can obtain by using the iteration is of the invariant linear combination $U+U'$. That is,

$$\begin{aligned}U(t,t_0)+U'(t,t_0) &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}dt^{(1)}\cdots\int_{t_0}^{t}dt^{(n)}\int_{-\infty}^{+\infty}d^3x^{(1)}\cdots\int_{-\infty}^{+\infty}d^3x^{(n)}\\ &\quad\times 2T(h_I(x^{(1)})\cdots h_I(x^{(n)})).\end{aligned}\tag{1}$$

Where, h_I is the interaction Hamiltonian density: $H_I(t)=\int d^3x\cdot h_I(x)$.

In order to define S matrices: $S\equiv S^{(+)}$ and $S'\equiv S^{(-)}$, we must modify Wick's theorem,[2] instead of respective usage of iteration method applied to U and U' separately. The ordinary Wick's theorem is unsatisfactory because the function $T(\cdots)$ in eq.(1), which is invariant under the frequency part exchange is expanded into the noninvariant function $N(\cdots)$. Thus we must expand $2T(\cdots)$ in eq.(1) into $N(\cdots)+N'(\cdots)$ which corresponds to

$$U(t,t_0)+U'(t,t_0)\to S^{(+)}+S^{(-)}=U(+\infty,-\infty)+U'(+\infty,-\infty).$$

Thus we shall obtain the following expression as the modified Wick's theorem:

$$2T(\phi_1\phi_2\cdots\phi_l)=s^{(+)}+s^{(-)},\tag{2}$$

where

$$s^{(+)}=\sum c\prod_{i=1}^{2m-1}(\underline{\phi_{k_i}\phi_{k_{i+1}}}\pm\overline{\phi_{k_i}\phi_{k_{i+1}}})N(\phi_{k_{2m+1}}\cdots\phi_{k_l}),$$

$$s^{(-)}=\sum c\prod_{i=1}^{2m-1}(\underline{\phi_{k_i}\phi_{k_{i+1}}}\pm\overline{\phi_{k_i}\phi_{k_{i+1}}})N'(\phi_{k_{2m+1}}\cdots\phi_{k_l}).$$

(Each term itself determines its signs in $\pm$, either $+$ or $-$. c is coefficient includes sign factor.)

Applying this modified formula to $2T\{h_I(x^{(1)})\cdots h_I(x^{(n)})\}$ in eq.(1), we may formally identify $S^{(+)}=U(+\infty,-\infty)$ and $S^{(-)}=U'(+\infty,-\infty)$ as

$$S^{(+)}=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}d^4x^{(1)}\cdots\int_{-\infty}^{+\infty}d^4x^{(n)}s^{(+)}(x^{(1)}\cdots x^{(n)}),\tag{3}$$

$$S^{(-)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} d^4x^{(1)} \cdots \int_{-\infty}^{+\infty} d^4x^{(n)} s^{(-)}(x^{(1)} \cdots x^{(n)}), \tag{4}$$

respectively. Thus we find that S matrix also has *both frequency parts*. In the following we only consider about the term $S^{(+)}$. The term $S^{(-)}$ is discarded. The reason of this discarding is explained briefly in section 6.

4. The extraction of self-energy terms

In the ordinary theory, when we extract the self-energy terms from the chronologically arranged function $T(\cdots)$, we have

$$2\underline{A_{\mu 1} A_{\nu 2}} \cdot \underline{\psi_1 \overline{\psi_2}}, \tag{5}$$

where $\underline{A_{\mu 1} A_{\nu 2}}$ and $\underline{\psi_1 \overline{\psi_2}}$ are the abbreviations of $\underline{A_\mu(x_1) A_\nu(x_2)}$ and $\underline{\psi(x_1)\overline{\psi}(x_2)}$, and these are the ordinary propagators of the photon and the electron respectively. On the other hand, when we shall follow the modified formulation explained in the previous section, we must extract the sum of two terms for the self-energy of a fermion:

$$\underline{A_{\mu 1} A_{\nu 2}} \cdot \overline{\psi_1 \overline{\psi_2}} + \overline{A_{\mu 1} A_{\nu 2}} \cdot \underline{\psi_1 \overline{\psi_2}}, \tag{6}$$

where we must take note of the formula

$$\underline{A_{\mu 1} A_{\nu 2}} \cdot \underline{\psi_1 \overline{\psi_2}} + \overline{A_{\mu 1} A_{\nu 2}} \cdot \overline{\psi_1 \overline{\psi_2}} = 0. \tag{7}$$

In order to derive (6) and (7), we must pick only the term $N(\cdots)$ up after expanding $2T(\cdots)$ into $N(\cdots) + N'(\cdots)$. The term $N'(\cdots)$ is discarded.
The formula (7) is the very reason why the divergence cancels in our formulation.
Thus we find the following correspondence:

The term (5) for the ordinary theory
corresponds to the term (6) for the modified theory.

This leads us to the following consequence about the Feynman integral:

$$\int_0^1 dx \cdots \textit{ should be replaced by } \int_{1-\kappa}^{\kappa} dx \cdots, \tag{8}$$

where x is the Feynman parameter and the κ is the arbitrary parameter within the range of $0 < \kappa < 1$.
The explanation about (6), (7) and the replacement (8) are given in the next section.

5. Self-energy of a fermion

When we intend to calculate the fermion self-energy we must deal the following type of the integral with which we may write formally

$$\int_{-\infty}^{\infty} d^4k \int_0^{\lambda^2} dL \frac{N}{A^2B} = \int_0^1 dx \int_0^{\lambda^2} dL \int_{-\infty}^{\infty} d^4k \frac{2xN}{[Ax + B(1-x)]^3}, \tag{9}$$

where the integral on L represents the Feynman cutoff integral.

In the following we denote $a = a_0 + i\epsilon, b = b_0 + i\epsilon, a' = a_0 - i\epsilon'$ and $b' = b_0 - i\epsilon'$.

(Both ϵ and ϵ' are infinitesimal positive quantity.)

Moreover following the Feynman's notation[3] we denote

$$N = -\frac{e^2}{\pi i}\gamma_\mu\{\gamma_\nu(p_\nu - k_\nu) + m\}\gamma_\mu, \quad a_0 = k^2 - L, \quad b_0 = k^2 - p \cdot k. \tag{10}$$

Taking into account the integlation contour on k_0 plane we find

$$\int_{-\infty}^{\infty} d^4k \frac{N}{[a_0x + b_0(1-x) - i\epsilon]^3} = -\int_{-\infty}^{\infty} d^4k \frac{N}{[a_0x + b_0(1-x) + i\epsilon]^3}. \tag{11}$$

This formula is verified by substituting $(a\ ,\ b)$ and $(a',\ b')$ into $(A\ ,\ B)$ in (9) respectively and perform the integral with respect to the energy momentum variable k_0 by using the Wick rotation in the right-hand side and *the reciprocal Wick rotation* in the left-hand side. Thus we get

$$\int_{-\infty}^{\infty} d^4k \frac{N}{a^2b} + \int_{-\infty}^{\infty} d^4k \frac{N}{a'^2b'} = 0. \tag{12}$$

This expression is identical to the formula (7) if we use the photon propagator with the Feynman cutoff.

To the contrary the integral (9) is divided into two terms when we integlate this on k_0, in the case that the denominator is a'^2b. In this case the denominator in the right-hand side in eq.(9) is written as

$$[a'x + b(1-x)]^3 = \left[a_0x + b_0(1-x) + i\epsilon\left(1 - \frac{1}{\kappa}x\right)\right]^3, \tag{13}$$

where $\kappa = \epsilon/(\epsilon + \epsilon')$. Thus substituting explicitly the adequate terms with the valid physical meaning into a' and b, we may find the integration contour about the energy momentum variable k_0 changes at the boundary where x is equal to κ.

Thus we have

$$\int_{-\infty}^{\infty} d^4k \frac{N}{a'^2b} = \int_{-\infty}^{\infty} d^4k \left\{ \int_0^{\kappa-\delta} \frac{2xdxN}{[a_0x + b_0(1-x)]^3} - \int_{\kappa+\delta}^1 \frac{2xdxN}{[a_0x + b_0(1-x)]^3} \right\}, \tag{14}$$

where δ is again the infinitesimal positive quantity. Afterwards we put $\delta = 0$.

The "$-$"sign appeared in the right-hand side represents the fact that the contour of k_0 integlation are different each other in both the first term and the second term. This equation is obtained by performing the Wick rotation in the first term and *the reciprocal Wick rotation* in the second term respectively.

Moreover we must add one more term in addition to the term in the left-hand side of eq.(14). Thus, as a whole, we must caluculate

$$\int_{-\infty}^{\infty} d^4k \frac{N}{a'^2 b} + \int_{-\infty}^{\infty} d^4k \frac{N}{a^2 b'}. \tag{15}$$

This lead us to

$$\int_{-\infty}^{\infty} d^4k \int_{1-\kappa}^{\kappa} \frac{2xdxN}{[a_0 x + b_0(1-x)]^3}, \tag{16}$$

where $1 - \kappa = \kappa' = \epsilon'/(\epsilon + \epsilon')$ is arbitrary within the range of $0 < \kappa < 1$.

The substitution of (10) into (16) and integrating on L from 0 to λ^2 gives

$$\frac{3me^2}{2\pi} \cdot dx \cdot \ln \frac{\lambda}{m}, \tag{17}$$

since the width of integral region takes infinitesimal amount dx itself at $x = 1/2$, as κ approaches to $1/2$.

Moreover we put $\lambda \gg 1$ relative to m and $\gamma_\mu p_\mu = m$.

This expression (17) is divergence-free although it is indefinite. We must add only dx as a factor up to the ordinary theory; x is the Feynman parameter which is cut both the upper and the lower boundaries off to bring dx into the final result (17).

Thus we must add *negative axis*, i.e. counter Fock space to ordinary Fock space to construct divergence-free quantum field theory. This procedure is inevitable.

6. Discussion

1.

$$< t|' \cdot |t > \equiv < t|'t > = < t|t >' = 0.$$

This expression is verified by rewriting this in matrix form. This is the reason why we discard the term $S^{(-)}$ as it's mentioned in section 3.

2. Results about the process without any loop coincide with ordinary theory.

This is obvious from the fact that in the case that any propagator is not a part of a loop,

$$\overline{AB} = \underline{AB}.$$

3. For the case of the boson self-energy the replacement (8) is also valid.

The validity of this replacement for the case of the boson self-energy is verified by using quite similar procedure as in the case of a fermion self-energy.

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The present method for solving a basic equation of quantum field theory is not in a proper way. In this paper we reconstruct the quantum field theory along the following program. As the most convenient illustration we deal with the most tipical quantum field theory: the quantum electorodynamics under consideration of the interaction picture. At first we introduce the transformation of *the freqency part exchange* to obtain the negative energy quantum field theory with the negatively divergent self-energy. After this theory was obtaied we may average self-energy terms out at the result obtained from two kinds of the theory: the ordinary theory with the positively divergent self-energy and the newly obtained theory with the negatively divergent self-energy. Thus divergence-free quantum field theory is obtainable. We are not able to afford any divergence-free quantum field theory without performing the procedure explained in the above.

The transformation of the frequency part exchange means that every field operator appearing in a theory is transformed each other into its opposite frequency part. Moreover it requires also the introduction of an adequate transformation of the Fock space. Furthermore, it leads us to the modification of the Wick's theorem. Thus, it is shown that the basic equation has two kinds of solution, i.e. the solution of Fock space and of *counter Fock space* (transformed Fock space under the transformation of the frequency part exchange). Finally It is shown as an example that theoretically calculated self-energy of a fermion is not a divergent quantity.