Wiener Functions of a Biharmonic Space

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1. Introduction

Let X be a connected, locally connected and locally compact Hausdorff space with a countable basis and $(X, \boldsymbol{H})$ be an elliptic biharmonic space in the sence of Smyrnēlis [9]. We denote by $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ the Brelot's harmonic spaces associated with $(X, \boldsymbol{H})$ and by $W^{(j)}(X)$ the totality of bounded continuous Wiener functions of $(X, \boldsymbol{H}_j)$ $(j=1, 2)$. In this note we consider analogously the totality of bounded continuous Wiener functions $W(X, \boldsymbol{H})$ of an elliptic biharmonic space $(X, \boldsymbol{H})$ (see § 3 for the definition) and give a characterization of $W(X, \boldsymbol{H})$ as follows.

Supposing that $(X, \boldsymbol{H})$ has the $\boldsymbol{H}_1$-Green's function and the constant function 1 is in $W^{(2)}(X)$, we shall show that the following six conditions are equivalent :
(1) $(0, 1)\in W(X, \boldsymbol{H})$; (2) $(0, e_2)$ is $\boldsymbol{H}$-harmonizable, where $e_2=h_1^{(2)}$;
(3) $\bar{W}(0, 1)\neq\phi$;
(4) $\int G_1(x, y)e_2(y)d\alpha(y)<+\infty$ for some $x\in X$, where $G_1(x, y)$ is the $\boldsymbol{H}_1$-Green's function of X and α is the composing measure of $(X, \boldsymbol{H})$;
(5) $\{f : (0, f)\in W(X, \boldsymbol{H})\}=W^{(2)}(X)$; (6) $W(X, \boldsymbol{H})=W^{(1)}(X)\times W^{(2)}(X)$.

Let (X_1^*, X_2^*) be a couple of two compactifications of X and (Δ_1, Δ_2) be a couple of their ideal boundaries(i. e. $\Delta_j=X_j-X$ $(j=1, 2)$. A couple (X_1^*, X_2^*) is called $\boldsymbol{H}$-resolutive if for any couple (f_1, f_2) of bounded continuous functions f_j on $\Delta_j(j=1, 2)$ there exists $(H_1(f_1, f_2), H_2(f_1, f_2))\in\boldsymbol{H}(X)$ with the boundary value (f_1, f_2). Supposing that there exists $(t_1, t_2)\in\boldsymbol{S}(X)$ with $\inf_{x\in X} t_j(x)>0$ $(j=1, 2)$ and the constant function 1 ia in $W^{(2)}(X)$, we shall show that (X_1^*, X_2^*) is $\boldsymbol{H}$-resolutive if and only if X_j^* is $\boldsymbol{H}_j$-resolutive $(j=1, 2)$. Hence in this case we know that the Riquier's boundary value problem on the ideal boundaries has an unique solution.

2. Biharmonic spaces

Let X be a connected,locally connected and locally compact Hausdorff space with a countable basis. For an open set $U=\phi$ in X, we denote by $\boldsymbol{C}(U)$ the real vector space of finite continuous functions on U. An element (h_1, h_2) in $\boldsymbol{C}(U)\times\boldsymbol{C}(U)$ is called compatible if $h_1=0$ on an open subset U' of U implies $h_2=0$ on U'. Let $\boldsymbol{H}$ be an application $U\rightarrow\boldsymbol{H}(U)$, where $\boldsymbol{H}(U)$ is a real vector subspace of compatible couples in $\boldsymbol{C}(U)\times\boldsymbol{C}(U)$. An ellement in $\boldsymbol{H}(U)$ is called $\boldsymbol{H}$-harmonic in U.

A relatively compact open set ω in X is called $\boldsymbol{H}$-regular if for any couple of (f_1, f_2) of finite continuous functions on the boundary $\partial\omega$ of ω, there exists a unique $(h_1, h_2)\in\boldsymbol{H}(\omega)$ such that:

(i) $\lim_{x\to a} h_j(x)=f_j(a)$ for any $a\in\partial\omega$ $(j=1, 2)$;
(ii) $f_j\geqq 0$ $(j=1, 2)$ implies $h_1\geqq 0$ and $f_2\geqq 0$ implies $h_2\geqq 0$.

For an $\boldsymbol{H}$-regular set ω, there exists a unique system $(\mu_x^\omega, \nu_x^\omega, \lambda_x^\omega)$ of positive Radon measures on $\partial\omega$ such that

$$h_1(x)=\int f_1 d\mu_x^\omega+\int f_2 d\nu_x^\omega,\ h_2(x)=\int f_2 d\lambda_x^\omega.$$

This system is called the system of biharmonic measures of $(X, \boldsymbol{H})$.

We say that $(X, \boldsymbol{H})$ is an elliptic biharmonic space in the sence of Smyrnélis [9] if it satisfies the following four axioms.

Axiom I. $\boldsymbol{H}$ is a sheaf on X.

Axiom II. The $\boldsymbol{H}$-regular open sets form a basis of X.

Let U be an open set in X. A couple (v_1, v_2) of functions on U is called $\boldsymbol{H}$-hyperharmonic on U if

(i) v_j is lower semi-continuous and $>-\infty$ on $U(j=1, 2)$,
(ii) $v_1(x)\geqq\int v_1 d\mu_x^\omega+\int v_2 d\nu_x^\omega$ and $v_2(x)\geqq\int v_2 d\lambda_x^\omega$ for any $\boldsymbol{H}$-regular neighborhood ω of x with $\bar{\omega}\subset U$.

The set of all $\boldsymbol{H}$-hyperharmonic couples on U is denoted by $\boldsymbol{H}^*(U)$. A couple $(s_1, s_2)\in\boldsymbol{H}^*(U)$ is called $\boldsymbol{H}$-superharmonic on U if s_j is not identically $+\infty$ on any connected component of U $(j=1, 2)$ and an $\boldsymbol{H}$-superharmonic couple (p_1, p_2) on U is called $\boldsymbol{H}$-potential on U if $p_j\geqq 0$ and, for any $(h_1, h_2)\in\boldsymbol{H}(U)$, $h_j=0$ so far as $0\leqq h_j\leqq p_j$ $(j=1, 2)$. The set of all $\boldsymbol{H}$-superharmonic couples (resp. $\boldsymbol{H}$-potentials) on U is denoted by $\boldsymbol{S}(U)$ (resp. $\boldsymbol{P}(U)$). For an open set U, we put $\boldsymbol{H}_1^*(U)$ $=\{v_1 : (v_1, 0)\in\boldsymbol{H}^*(U)\}$, $\boldsymbol{H}_2^*(U)=\{v_2 : (v_1, v_2)\in\boldsymbol{H}^*(U)$ for some $v_1\}$, and $\boldsymbol{H}_j(U)$ $=\boldsymbol{H}_j^*(U)\cap[-\boldsymbol{H}_j^*(U)]$ $(j=1, 2)$.

Axiom III. (i) $\boldsymbol{H}_j^*(X)$ separates the points of X linearly $(j=1, 2)$.
(ii) On each relatively compact open set U there exists a strictly positive $h_j\in\boldsymbol{H}_j(U)$ $(j=1, 2)$.

Axiom IV. If U is a domain in X and $\{h_j^{(n)}\}_n$ is an increasing sequence of functions in $\boldsymbol{H}_j(U)$, then either $\sup_n h_j^{(n)}=+\infty$ or $\sup_n h_j^{(n)}\in\boldsymbol{H}_j(U)$ $(j=1, 2)$.

Set $\boldsymbol{H}_j=\{\boldsymbol{H}_j(U) : U$ is open set in $X\}$. It is shown by Theorem 1.29 in [9] that $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ is a Brelot's harmonic space. We call $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ the Brelot's harmonic space associated with $(X, \boldsymbol{H})$. The set of all $\boldsymbol{H}$-superharmonic functions (resp. $\boldsymbol{H}_j$-potentials) on U is denoted by $\boldsymbol{S}_j(U)$(resp. $\boldsymbol{P}_j(U)(j=1, 2)$.

Let $(X, \boldsymbol{H})$ be an ellptic biharmonic space and $(X, \boldsymbol{H}_j)$ $(j=1, 2)$ be the Brelot's harmonic space associated with $(X, \boldsymbol{H})$. Denote by Ω the set of all $\boldsymbol{H}$-regular sets in X. We say that $(X, \boldsymbol{H}_1)$ has a consistent system $\{G_1^\omega(x, y) : \omega \in \Omega\}$ of $\boldsymbol{H}_1$-Green's functions if to each $\omega \in \Omega$ there corresponds a function $G_1^\omega(x, y)$ on $\omega \times \omega$ having the following properties:

(i) for each $y \in \omega$, $G_1^\omega(\cdot, y)$ is an $\boldsymbol{H}_1$-potential on ω and $\boldsymbol{H}_1$-harmonic on $\omega - \{y\}$,

(ii) if $\omega' \subset \omega$, $\omega' \in \Omega$ and $y \in \omega'$ then the function $G_1^\omega(x, y) - G_1^{\omega'}(x, y)$ of x is $\boldsymbol{H}_1$-harmonic on ω';

(iii) for each $\boldsymbol{H}_1$-potential p on ω, there exists a unique positive Radon measure β on ω such that $p(x) = \int G_1^\omega(x, y) d\beta(y)$.

By Theorem 9 in [13] we have

Lemma 1. *Let $(X, \boldsymbol{H})$ be an elliptic biharmonic space. Suppose that $(X, \boldsymbol{H}_1)$ has a consistent system $\{G_1^\omega(x, y) : \omega \in \Omega\}$ of $\boldsymbol{H}_1$-Green's functions. Then there exists a unique positive Radon measure α on X such that*

$$\nu_x^\omega = \int G_1^\omega(x, y) \lambda_y^\omega d\alpha(y)$$

for any $\omega \in \Omega$ and any $x \in \omega$, that is for any finite continuous function f on X

$$\int f d\nu_x^\omega = \int G_1^\omega(x, y) \left(\int f d\lambda_y^\omega \right) d\alpha(y).$$

This positive Radon measure α is called the **composing measure** of $(X, \boldsymbol{H})$. By virtue of the compatibility of biharmonic couples, this measure is everywhere dense in X.

A biharmonic space $(X, \boldsymbol{H})$ is called **strict** if it satisfies the following axiom.

Axiom (P) : For any point $x \in X$, there exists $(p_1, p_2) \in \boldsymbol{P}(X)$ such that $p_j(x) > 0$ $(j = 1, 2)$.

From now on we suppose that a biharmonic space $(X, \boldsymbol{H})$ is strict. Let E be a closed set in X and $(u_1, u_2) \in \boldsymbol{H}^*(X)^+$. We put $(R_1^E(u_1, u_2), R_2^E(u_1, u_2) = (\inf v_1, \inf v_2)$ where $(v_1, v_2) \in \boldsymbol{H}^*(X)^+$ and $v_j \geqq u_j$ on $E (j = 1, 2)$. We denote by $(\hat{R}_1^E(u_1, u_2), \hat{R}_2^E(u_1, u_2))$ the lower semi-continuous regularization of $(R_1^E(u_1, u_2), R_1^E(u_1, u_2))$. By Corollary 5.7 in [9], $(\hat{R}_1^E(u_1, u_2), \hat{R}_1^E(u_1, u_2))$ is $\boldsymbol{H}$-hyperharmonic on X and $\boldsymbol{H}$-harmonic on $X - E$.

For an open set $U \subset X$ and a couple (f_1, f_2) of real valued functions on the boundary ∂U of U, we denote by $\overline{\boldsymbol{V}}^U(f_1, f_2)$ the set of lower bounded couples $(v_1, v_2) \in \boldsymbol{H}^*(U)$ such that non-negative outside a compact set of X and

$$\liminf_{x \to a} v_j(x) \geqq f_j(a) \ (j = 1, 2)$$

for any $a \in \partial U$ and we put $\underline{\boldsymbol{V}}^U(f_1, f_2) = \{(v_1, v_2) : (-v_1, -v_2) \in \overline{\boldsymbol{V}}^U(-f_1, -f_2)\}$.
If $\overline{\boldsymbol{V}}^U(f_1, f_2)$ and $\underline{\boldsymbol{V}}^U(f_1, f_2)$ are both non-empty, we put

$$(\overline{H}_1^U(f_1, f_2), \overline{H}_2^U(f_1, f_2)) = (\inf v_1, \inf v_2) \text{ where } (v_1, v_2) \in \overline{\boldsymbol{V}}^U(f_1, f_2),$$
$$(\underline{H}_1^U(f_1, f_2), \underline{H}_2^U(f_1, f_2)) = (\sup v_1, \sup v_2) \text{ where } (v_1, v_2) \in \underline{\boldsymbol{V}}^U(f_1, f_2),$$

Then by Corollary 5.7 in [9], $(\overline{H}_1^U(f_1, f_2), \overline{H}_2^U(f_1, f_2))$ and $(\underline{H}_1^U(f_1, f_2), \underline{H}_2^U(f_1, f_2))$ are

$\boldsymbol{H}$-harmonic on U and $\overline{H}_j(f_1, f_2) \geqq \underline{H}_j(f_1, f_2)$ $(j=1, 2)$.
If$(\overline{H}_1^U(f_1, f_2), \overline{H}_2^U(f_1, f_2))=(\underline{H}_1^U(f_1, f_2), \underline{H}_2^U(f_1, f_2))$ we denote by $(H_1(f_1, f_2), H_2(f_1, f_2))$ the common couple.

Lemma 2. *Let U be an open set in X. Then for any $(u_1, u_2) \in \boldsymbol{H}^*(X)^+$, we have $(\hat{R}_1^{X-U}(u_1, u_2), \hat{R}_2^{X-U}(u_1, u_2))=(\overline{H}_1^U(u_1, u_2), \overline{H}_2^U(u_1, u_2))$ on U.*

Proof. Let $(v_1, v_2) \in \overline{\boldsymbol{V}}^U(u_1, u_2)$ and

$$(w_1, w_2)=\begin{cases} ((u_1, u_2) & \textit{on } X-U \\ (\inf(u_1, v_1), \inf(u_2, v_2) & \textit{on } U \end{cases}$$

Then $(w_1, w_2) \in \boldsymbol{H}^*(X)^+$ and $(w_1, w_2) \geqq (u_1, u_2)$ on $X-U$. Hence $(\hat{R}_1^{X-U}(u_1, u_2), \hat{R}_2^{X-U}(u_1, u_2)) \leqq (v_1, v_2)$ on U and so we have

$$(\hat{R}_1^{X-U}(u_1, u_2), \hat{R}_2^{X-U}(u_1, u_2) \leqq (\overline{H}_1^U(u_1, u_2), \overline{H}_2^U(u_1, u_2)).$$

Conversely for any $(v_1, v_2) \in \boldsymbol{H}^*(X)^+$ satifing $(v_1, v_2) \geqq (u_1, u_2)$ on $X-U$, we have $(v_1, v_2) \in \overline{\boldsymbol{V}}^U(u_1, u_2)$. Therefore we get

$$(\hat{R}_1^{X-U}(u_1, u_2), \hat{R}_2^{X-U}(u_1, u_2) \geqq (\overline{H}_1^U(u_1, u_2), \overline{H}_2^U(u_1, u_2)).$$

Lemma 3. *Let be a compact set of X and $(s_1, s_2) \in \boldsymbol{S}(X)^+$.*
Then $(\hat{R}_1^K(s_1, s_2), \hat{R}_2^K(s_1, s_2)) \in \boldsymbol{P}(X)$.

Proof. Let L be a compact subset satisfing $K \subset L$. Since $(\hat{R}_1^K(s_1, s_2), \hat{R}_2^K(s_1, s_2))$ is $\boldsymbol{H}$-harmonic on $X-K$ and $(X, \boldsymbol{H})$ is strict, there exists an $\boldsymbol{H}$-potential (p_1, p_2) such that $\hat{R}_j^E(s_1, s_2) \leqq p_j$ on ∂L $(j=1, 2)$. Since

$$(\hat{R}_1^K(s_1, s_2), \hat{R}_2^K(s_1, s_2))=(\overline{H}_1^{X-K}(s_1, s_2), \overline{H}_2^{X-K}(s_1, s_2))$$

on $X-K$ and

$$\begin{aligned}&(\overline{H}_1^{X-K}(s_1, s_2), \overline{H}_2^{X-K}(s_1, s_2)) \\ &=(\overline{H}_1^{X-L}(\hat{R}_1^K(s_1, s_2), \hat{R}_2^K(s_1, s_2)), \overline{H}_2^{X-L}(\hat{R}_1^K(s_1, s_2), \hat{R}_2^K(s_1, s_2))\end{aligned}$$

on $X-L$, we have $(\hat{R}_1^K(s_1, s_2), \hat{R}_2^K(s_1, s_2) \leqq (p_1, p_2)$ on $X-L$.
Hence $(\hat{R}_1^K(s_1, s_2), \hat{R}_2^K(s_1, s_2)) \in \boldsymbol{P}(X)$.

3. Wiener functions of $(X, \boldsymbol{H})$

Let (f_1, f_2) be a couple of real valued functions on X. We set $\overline{\boldsymbol{W}}(f_1, f_2) =\{(s_1, s_2) \in \boldsymbol{S}(X) : s_j \geqq f_j$ on for some compact set K in X $(j=1, 2)\}$,
$\underline{\boldsymbol{W}}(f_1, f_2)=\{(s_1, s_2) : (-s_1, -s_2) \in \overline{\boldsymbol{W}}(-f_1, -f_2)\}$. If $\overline{\boldsymbol{W}}(f_1, f_2) \neq \phi$ and $\underline{\boldsymbol{W}}(f_1, f_2) \neq \phi$, we put
$(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2))=\{\inf s_1, \inf s_2)$ where $(s_1, s_2) \in \overline{\boldsymbol{W}}(f_1, f_2)$,
$(\underline{h}_1(f_1, f_2), \underline{h}_2(f_1, f_2))=(\sup s_1, \sup s_2)$ where $(s_1, s_2) \in \underline{\boldsymbol{W}}(f_1, f_2)$.
Then by Theorem 5.6 in [9] we have $(\overline{h}_1(f_1, f_2)), \overline{h}_2(f_1, f_2)) \in \boldsymbol{H}(X)$,
$(\underline{h}_1(f_1, f_2), \underline{h}_2(f_1, f_2)) \in \boldsymbol{H}(X)$ and $(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2)) \geqq (\underline{h}_1(f_1, f_2), \underline{h}_2(f_1, f_2))$.

Lemma 4. *If $\overline{W}(f_1, f_2)\neq\phi$ and $\underline{W}(f_1, f_2)\neq\phi$, then there exists $(p_1, p_2)\in \boldsymbol{P}(X)$ and for any $\varepsilon>0$*

$$(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2))+\varepsilon(p_1, p_2)\in \overline{W}(f_1, f_2)$$
$$(\underline{h}_1(f_1, f_2), \underline{h}_2(f_1, f_2))-\varepsilon(p_1, p_2)\in \underline{W}(f_1, f_2).$$

Proof. We take $(s_1^{(n)}, s_2^{(n)})\in \overline{W}(f_1, f_2)$ such that

$$\sum_{n=1}^{\infty}(s_1^{(n)}-\overline{h}_1(f_1, f_2), s_2^{(n)}-\overline{h}_2(f_1, f_2))$$

is convergent at a point. Let K_n be a compact set such that $s_j^{(n)}\geqq f_j$ on $X-K_n$ $(j=1, 2)$. Let $\{X_n\}$ be an exhaustion of X satisfing $K_i\subset X_n$ for $i\leqq 2n$. We put $p_j^{(n)}=R_j^{\bar{X}_n}(s_1^{(n)}-\overline{h}_1(f_1, f_2), s_2^{(n)}-\overline{h}_2(f_1, f_2))$ $(j=1, 2)$. By Lemma 3, $(p_1^{(n)}, p_2^{(n)})$ being an $\boldsymbol{H}$-potential on X, $(p_1, p_2)=\sum_{n=1}^{\infty}(p_1^{(n)}, p_2^{(n)})$ is an $\boldsymbol{H}$-potential on X. For any $\varepsilon>0$ we take m satisfing $m\geqq\frac{1}{\varepsilon}$, then for any $k\geqq 1$ and $x\in X_{m+2k}-K_{m+k}$,

$$\overline{h}_j(f_1, f_2)(x)+\varepsilon p_j(x)\geqq \overline{h}_j(f_1, f_2)(x)+\frac{1}{m}\sum_{n=m+2k}^{2m+2k}p_j^{(n)}(x)$$
$$=\frac{1}{m}\sum_{n=m+2k}^{2m+2k}s_j^{(n)}(x)\geqq f_j(x)\ (j=1, 2).$$

Since $(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2))+\varepsilon(p_1, p_2)\geqq(f_1, f_2$ on $X-X_{m+1}$, we have
$$(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2))+\varepsilon(p_1, p_2)\in \overline{W}(f_1, f_2).$$

Definition 1. *A couple (f_1, f_2) of real valued functions on X is called $\boldsymbol{H}$-harmoni-zable if $(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2))=(\underline{h}_1(f_1, f_2), \underline{h}_2(f_1, f_2))$. In this case we denote by $(h_1(f_1, f_2), h_2(f_1, f_2))$ the common couple.*

Let (s_1, s_2) be an $\boldsymbol{H}$-superharmonic couple on X with an $\boldsymbol{H}$-subharmonic minorant couple. Then (s_1, s_2) is $\boldsymbol{H}$-harmonizable and $(h_1(s_1, s_2), h_2(s_1, s_2))$ is the greatest $\boldsymbol{H}$-harmonic minorant of (s_1, s_2).

Theorem 1. *If (f_1, f_2) and (g_1, g_2) are $\boldsymbol{H}$-harmonizable,then $\alpha(f_1, f_2)+\beta(g_1, g_2)$ $(\alpha, \beta\in \boldsymbol{R})$, $(\max(f_1, g_1), \max(f_2, g_2)$ and $(\min(f_1, g_1)), \min(f_2, g_2))$ are $\boldsymbol{H}$-harmonizable. In this case we have*
$h_j(\alpha f_1+\beta g_1, \alpha f_2+\beta g_2)=\alpha h_j(f_1, f_2)+\beta h_j(g_1, g_2)(j=1, 2)$,
$(h_1(\max(f_1, g_1), \max(f_2, g_2)), h_2(\max(f_1, g_1), \max(f_2, g_2)))$ *is the least $\boldsymbol{H}$-harmonic majorant of* $(h_1(f_1, f_2), h_2(f_1, f_2))$ *and* $(h_1(g_1, g_2), h_2(g_1, g_2))$ *and*
$(h_1(\min(f_1, g_1), \min(f_2, g_2)), h_2(\min(f_1, g_1), \min(f_2, g_2)))$ *is the greatest $\boldsymbol{H}$-harmonic minorant of* $(h_1(f_1, f_2), h_2(f_1, f_2))$ *and* $(h_1(g_1, g_2), h_2(g_1, g_2))$.

Proof. By Lemma 4 it is easily shown that $(f_1, f_2)+(g_1, g_2)$, $\alpha(f_1, f_2)(\alpha\geqq 0)$ and $(-f_1, -f_2)$ are $\boldsymbol{H}$-harmonizable. Hence $\alpha(f_1, f_2)+\beta(g_1, g_2)$ is $\boldsymbol{H}$-harmonizable and $h_j(\alpha f_1+\beta g_1, \alpha f_2+\beta g_2)=\alpha h_j(f_1, f_2)+\beta h_j(g_1, g_2)(j=1, 2)$.
Since $(h_1(f_1, f_2), h_2(f_1, f_2))$ and $(h_1(g_1, g_2), h_2(g_1, g_2))$ are expressed by the difference of non-negative $\boldsymbol{H}$-harmonic couples, there exists their least $\boldsymbol{H}$-harmonic majorant (u_1, u_2). By Lemma 4, there exist $(p_1, p_2), (q_1, q_2)\in \boldsymbol{P}(X)$ such that

$$(h_1(f_1, f_2), h_2(f_1, f_2))+\varepsilon(p_1, p_2)\in \overline{W}(f_1, f_2)$$
$$(h_1(g_1, g_2), h_2(g_1, g_2))+\varepsilon(q_1, q_2)\in \overline{W}(g_1, g_2)$$

Then $(u_1, u_2)+\varepsilon(p_1, p_2)+\varepsilon(q_1, q_2)\in \overline{W}(\max(f_1, g_1), \max(f_2, g_2))$ and so we have $\overline{h}_j(\max(f_1, g_1), \max(f_2, g_2))\leqq u_j(j=1, 2)$. On the other hand, we have $\underline{h}_j(\max(f_1, g_1), \max(f_2, g_2))\geqq h_j(f_1, f_2), h_j(g_1, g_2)(j=1, 2)$ and so $h_j(\max(f_1, g_1), \max(f_2, g_2))\geqq u_j(j=1, 2)$. Hence $(\max(f_1, g_1), \max(f_2, g_2))$ is $\boldsymbol{H}$-harmonizable and $h_j(\max(f_1, g_1), \max(f_2, g_2))=u_j(j=1, 2)$.

We denote by $\boldsymbol{HP}(X)$ the totality of all $\boldsymbol{H}$-harmonic couples on X which are differences of non-negative $\boldsymbol{H}$-harmonic couples, by $\widetilde{\boldsymbol{W}}(X, \boldsymbol{H})$ the totality of all $\boldsymbol{H}$-harmonizable couples and by $\widetilde{\boldsymbol{W}}_0(X, \boldsymbol{H})$ the totality of all couples $(f_1, f_2)\in \widetilde{\boldsymbol{W}}(X, \boldsymbol{H})$ satisfing $h_j(f_1, f_2)=0$ $(j=1, 2)$.

By Lemma 4, we have the following characterization of (f_1, f_2) in $\widetilde{\boldsymbol{W}}_0(X, \boldsymbol{H})$.

Lemma 5. *A couple (f_1, f_2) is in $\widetilde{\boldsymbol{W}}(X, \boldsymbol{H})$ if and only if there exists $(p_1, p_2)\in \boldsymbol{P}(X)$ such that $|f_j|\leqq p_j(j=1, 2)$ outside a compact set in X.*

Lemma 6. *If $(0, f)\in \widetilde{\boldsymbol{W}}(X, \boldsymbol{H})$ and $f\geqq 0$, then $h_1(0, f)$ is a continuous $\boldsymbol{H}_1$-potential on X and we have $h_1(0, f)(x)=\int G_1(x, y)h_2(0, f)d\alpha(y)$.*

Proof. Since $(0, f)\in \widetilde{\boldsymbol{W}}(X, \boldsymbol{H})$ and $f\geqq 0$, $(h_1(0, f), h_2(0, f))$ is a non-negative $\boldsymbol{H}$-harmonic couple on X and so $h_1(0, f)\in \boldsymbol{S}_1(X)^+$. By Lemma 5, there exists (p_1, p_2) $\in \boldsymbol{P}(X)$ such that for any $\varepsilon>0$ $(h_1(0, f), h_2(0, f))-\varepsilon(p_1, p_2)\in \underline{\boldsymbol{W}}(0, f)$. Hence $h_1(0, f)$ $\leqq \varepsilon p_1$ outside a compact set in X and so $h_1(0, f)\in \boldsymbol{P}_1(X)$. The integral representaion is shown by Lemma 1.

Theorem 2. *We have the following direct decomposition*:

$$\widetilde{\boldsymbol{W}}(X, \boldsymbol{H})=H\boldsymbol{P}(X)\oplus \widetilde{\boldsymbol{W}}_0(X, \boldsymbol{H}).$$

Let $(X, \boldsymbol{H}_j)(J=1, 2)$ be the Brelot's harmonic space associated with $(X, \boldsymbol{H})$. For a real-valued function on X, we set $\overline{\boldsymbol{W}}_f^{(j)}=\{s\in \boldsymbol{S}_j(X) : s\geqq f$ outside a compact set$\}$ and $\underline{\boldsymbol{W}}_f^{(j)}=\{s : -s\in \overline{\boldsymbol{W}}_{-f}^{(j)}\}$. If $\overline{\boldsymbol{W}}_f^{(j)}\neq \phi$, $\underline{\boldsymbol{W}}_f^{(j)}\neq \phi$, we define $\overline{h}_f^{(j)}=inf\overline{\boldsymbol{W}}_f^{(j)}$ and $\underline{h}_f^{(j)}=sup\underline{\boldsymbol{W}}_f^{(j)}$. We say that f is $\boldsymbol{H}_j$-harmonizable if $\overline{h}_f^{(j)}=\underline{h}_f^{(j)}$ and denote the common value by $h_f^{(j)}$. A function f on X is called a Wiener function of $(X, \boldsymbol{H}_j)$ if f is bounded continuous and $\boldsymbol{H}_j$-harmonizable. The totaity of Wiener functions of $(X, \boldsymbol{H}_j)$ is denoted by $\boldsymbol{W}^{(j)}(X)$. A function f is called a Wiener potential of $(X, \boldsymbol{H}_j)$ if $f\in \boldsymbol{W}^{(j)}(X)$ and $h_f^{(j)}=0$. The totality of Wiener potentials of $(X, \boldsymbol{H}_j)$ is denoted by $\boldsymbol{W}_0^{(j)}(X)(j=1, 2)$. Similarly we shall give the definition of Wiener functions of $(X, \boldsymbol{H})$ as follows.

Definition 2. *A couple of bounded continuous $\boldsymbol{H}$-harmonizable functions is called a Wiener function of $(X, \boldsymbol{H})$. The totality of Wiener functions of $(X, \boldsymbol{H})$ is denoted by $\boldsymbol{W}(X, \boldsymbol{H})$. A couple $(f_1, f_2)\in \boldsymbol{W}(X, \boldsymbol{H})$ is called a Wiener potential of $(X, \boldsymbol{H})$ if it satisfies $h_j(f_1, f_2)=0$ $(j=1, 2)$. The totality of Wiener potentials of $(X, \boldsymbol{H})$ is denoted by $\boldsymbol{W}_0(X, \boldsymbol{H})$.*

From the definition it follows immediately that $\{f : (f, 0)\in \boldsymbol{W}(X, \boldsymbol{H})\}=\boldsymbol{W}^{(1)}(X)$, $\{f : (0, f)\in \boldsymbol{W}(X, \boldsymbol{H})\}\subset \boldsymbol{W}^{(2)}(X)$ and $h_1(f, 0)=h_f^{(1)}$, $h_2(0, f)=h_f^{(2)}$. Further we have

Lemma 7. $\boldsymbol{W}(X, \boldsymbol{H})\subset \boldsymbol{W}^{(1)}(X)\times \boldsymbol{W}^{(2)}(X)$ *and* $\boldsymbol{W}_0(X, \boldsymbol{H})\subset \boldsymbol{W}_0^{(1)}(X)\times \boldsymbol{W}_0^{(2)}(X)$.

Proof. Let $(f_1, f_2)\in \boldsymbol{W}(X, \boldsymbol{H})$ and $f_j^+=\max(f, 0)$, $f_j^-=\max(-f, 0)(j=1, 2)$. Then $(f_1, f_2)=(f_1^+, f_2^+)-(f_1^-, f_2^-)$ and (f_1^+, f_2^+), $(f_1^-, f_2^-)\in \boldsymbol{W}(X, \boldsymbol{H})$ by Theorem 1. Hence we may suppose that $f_j\geqq 0$ $(j=1, 2)$. By Theorem 2, $(f_1, f_2)=(h_1(f_1, f_2), h_2(f_1, f_2))+(g_1, g_2)$ with $(g_1, g_2)\in \tilde{\boldsymbol{W}}_0(X, \boldsymbol{H})$. By Lemma 5 and Corollaire 5.16 in [9], we know that g_j is $\boldsymbol{H}_j$-harmonizable and $h_{g_j}^{(j)}=0$ $(j=1, 2)$. Since $h_1(f_1, f_2)$(resp. $h_2(f_1, f_2)$) is a nonnegative $\boldsymbol{H}_1$-superharmonic(resp. $\boldsymbol{H}_2$-harmonic) function, $h_1(f_1, f_2)$ is $\boldsymbol{H}_1$-harmonizable (resp. $h_2(f_1, f_2)$ is $\boldsymbol{H}_2$-harmonizable). Hence we have $f_j=h_j(f_1, f_2)+g_j\in \boldsymbol{W}^{(j)}(X)(j=1, 2)$.

From now on we suppose that $(X, \boldsymbol{H}_1)$ has a consistent system $\{G_1^{\omega}(x, y) : \omega\in\Omega\}$ of $\boldsymbol{H}_1$-Green's functions. Then $(X, \boldsymbol{H}_1)$ being strict, there exists an $\boldsymbol{H}_1$-Green's function $G_1(x, y)$ on X such that $G_1^{\omega}(x, y)=G_1(x, y)-\int G_1(z, y)d\mu_x^{\omega}(z)$ for any $\omega\in\Omega$ and there exists the composing measure α of $(X, \boldsymbol{H})$ by Lemma 1.

As for the inverse inclusion of Lemma 7 we have the following

Theorem 3. *Suppose that* $1\in \boldsymbol{W}^{(2)}(X)$ *and put* $e_2=h_1^{(2)}$. *Then the following* (1), (2), (3), (4), (5) *and* (6) *are equivalent:*

(1) $(0, 1)\in \boldsymbol{W}(X, \boldsymbol{H})$;
(2) $(0, e_2)$ *is* $\boldsymbol{H}$-*harmonizable*;
(3) $\bar{\boldsymbol{W}}(0, 1)\neq\phi$
(4) $\int G_1(x, y)e_2(y)d\alpha(y)<+\infty$ *for some* $x\in X$;
(5) $\{f : (0, f)\in \boldsymbol{W}(X, \boldsymbol{H})\}=\boldsymbol{W}^{(2)}(X)$;
(6) $\boldsymbol{W}(X, \boldsymbol{H})=\boldsymbol{W}^{(1)}(X)\times \boldsymbol{W}^{(2)}(X)$.

Proof.(1)$\Rightarrow$(2) There exists $(s_1, s_2)\in \boldsymbol{S}(X)^+$ such that $s_2\geqq 1$ outside a compact set. Then $s_2\geqq h_1^{(2)}=e_2$ and so $(s_1, e_2)\in \boldsymbol{S}(X)^+$. Since (s_1, e_2) and $(s_1, 0)$ are $\boldsymbol{H}$-harmonizable, $(0, e_2)=(s_1, e_2)-(s_1, 0)$ is $\boldsymbol{H}$-harmonizable. (1)$\Rightarrow$(3) This is evident from the definition. (2)$\Rightarrow$(4) Since $(0, e_2)$ is $\boldsymbol{H}$-harmonizble,$(h_1(0, e_2), h_2(0, e_2))=(q_1, e_2)\in \boldsymbol{H}(X)$ and by Lemma 6 $h_1(0, e_2)=q_1$ is continuous $\boldsymbol{H}_1$-potential on X. Therefore we have $q_1(x)=\int G_1(x, y)e_2(y)d\alpha(y)$ by Lemma 1 and so we have (3). It is easy to show that (4)$\Rightarrow$(2). (2)$\Rightarrow$(5) Since $\{f : (0, f)\in \boldsymbol{W}(X, \boldsymbol{H})\subset \boldsymbol{W}^{(2)}(X)$, it suffices to show the inverse inclusion. Let f be in $\boldsymbol{W}^{(2)}(X)$. By Theorem 2.2 in [6] we may assume that $0\leqq f\leqq 1$. We put $g=f-h_f^{(2)}$, then g is $\boldsymbol{H}_2$-harmonizable and $h_g^{(2)}=0$. Since $\bar{\boldsymbol{W}}(0, e_2)\neq\phi$, there exists $(s_1, s_2)\in \bar{\boldsymbol{W}}(0, e_2)$. Then $s_2\geqq h_f^{(2)}$ and so $(s_1, h_f^{(2)})\in \boldsymbol{S}(X)^+$. Since $(s_1, h_f^{(2)})$ and $(s_1, 0)$ are $\boldsymbol{H}$-harmonizable, $(0, h_f^{(2)})$ is $\boldsymbol{H}$-harmonizable. By Lemma 2.2 in [6] there exists $p_2\in \boldsymbol{P}_2(X)$ such that $|g|\leqq p_2$ outside a compact set. Denote by p_1 the $\boldsymbol{H}_1$-potential part of s_1. Then $(2p_1, \min(2s_2, p_2))\in \boldsymbol{P}(X)$ by Corollary 5.16 in [9] and $(0, |g|)\leqq(2p_1, \min(2s_2, p_2))$. Hence $(0, g)$ is $\boldsymbol{H}$-harmonizable by Lemma 5. Therefore $(0, f)=(0, h_f^{(2)})+(0, g)$ is $\boldsymbol{H}$-harmonizable and so $(0,f)\in \boldsymbol{W}(X, \boldsymbol{H})$. (3)$\Rightarrow$(5) This is shown similarly to the above proof. (5)$\Rightarrow$(6) Since $\boldsymbol{W}(X, \boldsymbol{H})\subset \boldsymbol{W}^{(1)}(X)\times \boldsymbol{W}^{(2)}(X)$ by Lemma 7, it suffices to show the inverse inclusion. Let $(f_1, f_2)\in \boldsymbol{W}^{(1)}(X)\times \boldsymbol{W}^{(2)}(X)$. Then $(f_1, 0)\in \boldsymbol{W}(X, \boldsymbol{H})$ and $(0, f_2)\in \boldsymbol{W}(X, \boldsymbol{H})$. Hence $(f_1, f_2)=(f_1, 0)+(0, f_2)\in \boldsymbol{W}(X, \boldsymbol{H})$. (6)$\Rightarrow$(1) Since $0\in \boldsymbol{W}^{(1)}(X)$ and $1\in \boldsymbol{W}^{(2)}(X)$, we have$(0, 1)\in \boldsymbol{W}(X, \boldsymbol{H})$.

Remark. In the cases (5) and (6) we have $\{f : (0, f)\in \boldsymbol{W}_0(X, \boldsymbol{H})\}=\boldsymbol{W}_0^{(2)}(X)$ and $\boldsymbol{W}_0(X, \boldsymbol{H})=\boldsymbol{W}_0^{(1)}(X)\times \boldsymbol{W}_0^{(2)}(X)$.

4. Resolutive compactifications of $(X, \boldsymbol{H})$.

Let $(X, \boldsymbol{H}_j)$ be the Brelot's harmonic space associated with $(X, \boldsymbol{H})$, X_j^* be a compactification of X and Δ_j be the ideal boundary of X_j^* $(j=1, 2)$. For a real valued function f on Δ_j, we denote by $\overline{V}_f^{(j)}$ the set of lower bounded $s_j\in \boldsymbol{S}_j(X)$ such that $\liminf_{x\to a} s_j(x)\geqq f(a)$ for any $a\in\Delta_j$ and we set $\underline{V}_f^{(j)}=\{s_j : -s_j\in \overline{V}_{-f}^{(j)}\}(j=1, 2)$.
If $\overline{V}_f^{(j)}$ and $\underline{V}_f^{(j)}$ are both non-empty, we put $\overline{H}_f^{(j)}=\inf \overline{V}_f^{(j)}$ and $\underline{H}_f^{(j)}=\sup \underline{V}_f^{(j)}$.
If $\overline{H}_f^{(j)}=\underline{H}_f^{(j)}$, f is called $\boldsymbol{H}_j$-resolutive and we denote by $H_f^{(j)}$ the common value $(j=1, 2)$.

For any $p_j\in \boldsymbol{P}_j(X)$ we put $\Gamma(p_j)=\{a\in\Delta_j : \liminf_{x\to a} p_j(x)=0\}$,
$\Gamma_j=\cap\Gamma(p_j)$ where $p_j\in \boldsymbol{P}_j(X)$ and $\Lambda_j=\Delta_j-\Gamma_j(j=1,2)$. A compactification X_j^* is called an $\boldsymbol{H}_j$-resolutive compactification if any bounded continuous function f on Δ_j is $\boldsymbol{H}_j$-resolutive $(j=1, 2)$. In this case , a point $a_j\in\Delta_j$ is called $\boldsymbol{H}_j$-regular if for any bounded continuous function f on Δ_j, $\lim_{x\to a_j} H_f^{(j)}(x)=f(a_j)$ $(j=1, 2)$. All points in Λ_j are not $\boldsymbol{H}_j$-regular $(j=1, 2)$.

Now we consider the following **condition (M. P.)**.

(M. P.) : There exists $(t_1, t_2)\in \boldsymbol{S}(X)$ such that $\inf_{x\in X} t_j(x)>0$ $(j=1, 2)$.

Lemma 8. (1) *If the constant function 1 is in $\boldsymbol{W}^{(2)}(X)$ and $(X, \boldsymbol{H})$ satisfies the condition (M. P.), then six conditions in Theorem 3 are satisfied.*

(2) *Conversely if the constant function 1 is in $\boldsymbol{W}^{(1)}(X)\cap \boldsymbol{W}^{(2)}(X)$ and one of six conditions in Theorem 3 is satisfied, then $(X, \boldsymbol{H})$ satisfies the condition **(M. P.)**.*

Proof. (1) Since $\overline{\boldsymbol{W}}(0, 1)\neq\phi$, this is trivial. (2) If the condition (1) in Theorem 3 is satisfied, then there exists $(s_1, s_2)\in \boldsymbol{S}(X)^+$ such that $(s_1, s_2)\geqq(0, 1)$ outside some compact set in X. Since $1\in \boldsymbol{W}^{(1)}(X)$,there exists $v_1\in \boldsymbol{S}_1(X)^+$ such that $v_1\geqq 1$ outside some compact set in X. Hence by virtue of Axiom (P), the condition **(M. P.)** is satisfied.

In this section we suppose that $(X, \boldsymbol{H})$ satisfies the condition **(M. P.)**. In this case we have the following minimum principle by Theorem 4.1 in [6].

Lemma 9. Let $(v_1, v_2)\in \boldsymbol{H}^*(X)$. If there exists $(p_1, p_2)\in \boldsymbol{P}(X)$ such that v_j+p_j is lower bounded and $\liminf_{x\to a} v_j(x)\geqq 0$ for any $a\in\Gamma_j$ $(j=1, 2)$,then $(v_1, v_2)\geqq(0, 0)$.

Let (X_1^*, X_2^*) be a couple of two compactifications of X and (Δ_1, Δ_2) be a couple of their ideal boundaries (i. e. $\Delta_j=X_j^*-X$ $(j=1, 2)$). For a couple (f_1, f_2) of real valued functions f_j defined on ideal boudaries Δ_j $(j=1, 2)$, we denote by $\overline{V}^*(f_1, f_2)$ the set of lower bounded couples $(s_1, s_2)\in \boldsymbol{S}(X)$ such that for any $a\in\Delta_j$

$$\liminf_{x\to a} s_j(x) \geqq f_j(a) \ (j=1,\ 2),$$

and we set $\underline{V}^*(f_1, f_2)=\{(s_1, s_2) : (-s_1, -s_2)\in \overline{V}^*(-f_1, -f_2)\}$. If $\overline{V}^*(f_1, f_2)$ and $\underline{V}^*(f_1, f_2)$ are both non-empty, we put

$$(\overline{H}_1(f_1,\ f_2),\ \overline{H}_2(f_1,\ f_2))=(\inf s_1,\ \inf s_2) \text{ where } (s_1,\ s_2)\in \overline{V}^*(f_1,\ f_2),$$
$$(\underline{H}_1(f_1,\ f_2),\ \underline{H}_2(f_1,\ f_2))=(\sup s_1,\ \sup s_2) \text{ where } (s_1,\ s_2)\in \underline{V}^*(f_1,\ f_2),$$

Then by Corollary 5.7 in [9],$(\overline{H}_1(f_1, f_2), \overline{H}_2(f_1, f_2))$ and $(\underline{H}_1(f_1, f_2), \underline{H}_2(f_1, f_2))$ are $\boldsymbol{H}$-harmonic on X. By Lemma 9, they are expressed by differeces of non-negative $\boldsymbol{H}$-harmonic couples and $(\overline{H}_1(f_1, f_2), \overline{H}_2(f_1, f_2))\geqq(\underline{H}_1(f_1, f_2), \underline{H}_2(f_1, f_2))$.
If $(\overline{H}_1(f_1, f_2), \overline{H}_2(f_1, f_2))=(\underline{H}_1(f_1, f_2), \underline{H}_2(f_1, f_2))$, (f_1, f_2) is called $\boldsymbol{H}$-resolutive and denote by $(H_1(f_1, f_2), H_2(f_1, f_2))$ the common couple. It is easly shown that if $(f, 0)$(resp. $(0, f)$) is $\boldsymbol{H}$-resolutive, then f is $\boldsymbol{H}_1$-resolutive (resp.$\boldsymbol{H}_2$-resolutive) and $H_1(f, 0)=H_f^{(1)}$ (resp. $H_2(0, f)=H_f^{(2)}$).

For any $(p_1, p_2)\in \boldsymbol{P}(X)$ we put

$$\Gamma(p_1,\ p_2)=\{(a_1,\ a_2)\in\Delta_1\times\Delta_2 : \liminf_{x\to a_j} p_j(x)=0 \ (j=1,\ 2)\}$$

and $\Gamma=\cap\Gamma(p_1, p_2)$ where $(p_1, p_2)\in \boldsymbol{P}(X)$. Then we have

Lemma 10. $\Gamma=\Gamma_1\times\Gamma_2$

Proof. Let $(a_1, a_2)\in\Gamma_1\times\Gamma_2$. For any $(p_1, p_2)\in \boldsymbol{P}(X)$, p_j being in $\boldsymbol{P}_j(X)$ $(j=1, 2)$ by Corollaire 5.13 in [9], we have $\liminf_{x\to a_j} p_j(x)=0$ $(j=1, 2)$ and so $(a_1, a_2)\in\Gamma$.
Hence we have $\Gamma\supset\Gamma_1\times\Gamma_2$. Conversely let (a_1, a_2) be any point in Γ. We take $p_j\in \boldsymbol{P}_j(X)$ $(j=1, 2)$. Then $(p_1, 0)$ being in $\boldsymbol{P}(X)$, $\liminf_{x\to a_1} p_1(x)=0$ and so $a_1\in\Gamma_1$. By the condition **(M. P.)**, there exists $(t_1, t_2)\in \boldsymbol{S}(X)$ such that $\inf_{x\in X} t_j(x)=c_j>0$ $(j=1, 2)$. Then we have $(t_1, \min(t_2, p_2))\in \boldsymbol{S}(X)^+$. Let $q_1=t_1-h_{t_1}^{(1)}$. Then we know that $(q_1, \min(t_2, p_2))\in \boldsymbol{S}(X)^+$, $q_1\in \boldsymbol{P}_1(X)$ and $\min(t_2, p_2)\in \boldsymbol{P}_2(X)$. Hence $(q_1, \min(t_2, p_2))\in \boldsymbol{P}(X)$ by Corollaire 5.16 in [9]. Therefore

$$\liminf_{x\to a_2} \min(t_2(x),\ p_2(x))=0.$$

Since $\liminf_{x\to a_2} \min(t_2(x), p_2(x))\geqq\min(c_2, \liminf_{x\to a_2} p_2(x))$, we have $\liminf_{x\to a_2} p_2(x)=0$ and so $a_2\in\Gamma_2$. Hence we know that $\Gamma\subset\Gamma_1\times\Gamma_2$.

By virtue of the condition **(M. P.)** we have the following

Theorem 4. *Let (f_1, f_2) be a couple of bounded continuous functions f_j on X_j^* $(j=1, 2)$. Then $(f_1, f_2)\in \boldsymbol{W}_0(X, \boldsymbol{H})$ if and only if $f_j=0$ on Γ_j $(j=1, 2)$.*

Proof. If $(f_1, f_2)\in \boldsymbol{W}_0(X, \boldsymbol{H})$, then there exists $(p_1, p_2)\in \boldsymbol{P}(X)$ such that $|f_j|\leqq p_j$ $(j=1, 2)$ outside a compact set. Hence $f_j=0$ on Γ_j $(j=1, 2)$. Conversely if $f_j=0$ on Γ_j $(j=1, 2)$, then f_j being in $\boldsymbol{W}_0^{(j)}(X)$ there exists $p_j\in \boldsymbol{P}_j(X)$ such that $|f_j|\leqq p_j$ $(j=1, 2)$ on X. By virtue of the condition **(M. P.)** there exists $(t_1, t_2)\in \boldsymbol{S}(X)$ such that $|f_j|\leqq t_j$ $(j=1, 2)$. Put $q_1=p_1+(t_1-h_{t_1}^{(1)})$ and $q_2=\min(p_2, t_2)$. Then $(q_1, q_2)\in \boldsymbol{P}(X)$ and $|f_j|\leqq q_j$ $(j=1, 2)$. Hence $(f_1, f_2)\in \boldsymbol{W}_0(X, \boldsymbol{H})$ by Lemma 5.

Similary to Lemma 4, we have

Lemma 11. *If $\overline{V}^*(f_1, f_2)$ and $\underline{V}^*(f_1, f_2)$ are both non-empty, there exists (s_1, s_2) $\in S(X)^+$ such that for any $\varepsilon>0$*

$$(\overline{H}_1(f_1, f_2), \overline{H}_2(f_1, f_2))+\varepsilon(s_1, s_2)\in \overline{V}^*(f_1, f_2),$$
$$(\underline{H}_1(f_1, f_2), \underline{H}_2(f_1, f_2))-\varepsilon(s_1, s_2)\in \underline{V}^*(f_1, f_2).$$

Using this lemma we have the following theorem similarly to the proof of Theorem 1.

Theorem 5. *If (f_1, f_2) and (g_1, g_2) are $\boldsymbol{H}$-resolutive, then $\alpha(f_1, f_2)+(g_1, g_2)(\alpha, \beta\in \boldsymbol{R})$, $(\max(f_1, g_1), \max(f_2, g_2))$ and $(\min(f_1, g_1), \min(f_2, g_2))$ are $\boldsymbol{H}$-resolutive and in this case we know that*
$H_j(\alpha f_1+\beta g_1, \alpha f_2+\beta g_2)=\alpha H_j(f_1, f_2)+\beta H_j(g_1, g_2)$ $(j=1, 2)$,
$(H_1(\max(f_1, g_1), \max(f_2, g_2)), H_2(\max(f_1, g_1), \max(f_2, g_2)))$ is the least $\boldsymbol{H}$-harmonic majorant of $(H_1(f_1, f_2), H_2(f_1, f_2))$ and $(H_1(g_1, g_2), H_2(g_1, g_2)$ and
$(H_1(min(f_1, g_1), min(f_2, g_2)), H_2(min(f_1, g_1), \min(f_2, g_2)))$ is the greatest $\boldsymbol{H}$-harmonic minorant of $(H_1(f_1, f_2), H_2(f_1, f_2))$ and $(H_1(g_1, g_2), H_2(g_1, g_2))$.

By this theorem, there exists a unique system of positive Radon measures $(\mu_x^*, \nu_x^*, \lambda_x^*)$ on ideal boundaries such that

$$H_1(f_1, f_2)(x)=\int f_1 d\mu_x^*+\int f_2 d\nu_x^*,\ H_2(f_1, f_2)(x)=\int f_2 d\lambda_x^*.$$

Where μ_x^* is a measure on Δ_1 and ν_x^* and λ_x^* are measures on Δ_2.

Lemma 12. *If $(0, f)$ is $\boldsymbol{H}$-resolutive and $f\geqq 0$, then $H_1(0, f)$ is a continuous $\boldsymbol{H}_1$-potential on X and we have*

$$H_1(0, f)(x)=\int G_1(x,y)H_2(0, f)(y)\,d\alpha(y)=\int G_1(x, y)\left(\int f d\lambda_y^*\right)d\alpha(y).$$

Proof. Since $(H_1(0, f), H_2(0, f))\in \boldsymbol{H}(X)$ and $H_j(0, f)\geqq 0$ $(j=1, 2)$ by Lemma 9, $H_1(0, f)$ is a non-negative continuous $\boldsymbol{H}_1$-superharmonic function on X. For any $\boldsymbol{H}_1$-harmonic minorant u_1 of $H_1(0, f)$, we have $u_1\leqq 0$ and so $H_1(0, f)$ is a continuous $\boldsymbol{H}_1$-potential on X. The integral representaion is shown by Lemma 1.

Lemma 13. *Let f_j be a bounded continuous function on X_j^* $(j=1, 2)$. Then (f_1, f_2) is $\boldsymbol{H}$-resolutive if and only if (f_1, f_2) is $\boldsymbol{H}$-harmonizable. In this case we have $(H_1(f_1, f_2), H_2(f_1, f_2))=(h_1(f_1, f_2), h_2(f_1, f_2))$.*

Proof. Since $\overline{W}(f_1, f_2)\subset \overline{V}^*(f_1, f_2)$, we have

$$(\overline{H}_1(f_1, f_2), \overline{H}_2(f_1, f_2))\leqq(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2)).$$

Let (t_1, t_2) be in $S(X)$ such that $\inf_{x\in X} t_j(x)>0$ $(j=1, 2)$. For any $(s_1, s_2)\in \overline{V}^*(f_1, f_2)$ and $\varepsilon>0$, $(s_1, s_2)+\varepsilon(t_1, t_2)$ being in $\overline{W}(f_1, f_2)$, $(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2))\leqq(s_1, s_2)+\varepsilon(t_1, t_2)$. Hence we have the inverse inequality

$$(\overline{H}_1(f_1, f_2), \overline{H}_2(f_1, f_2))\geqq(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2)).$$

Therefore we get $(\overline{H}_1(f_1, f_2), \overline{H}_2(f_1, f_2))=(\overline{h}_1(f_1, f_2), \overline{h}_2(f_1, f_2))$. Similary we have the relation $(\underline{H}_1(f_1, f_2), \underline{H}_2(f_1, f_2))=(\underline{h}_1(f_1, f_2), \underline{h}_2(f_1, f_2))$.

Definition 3. *A couple (X_1^*, X_2^*) of two compactifications of X is called $\boldsymbol{H}$-resolutive compactification if for any of bounded continuous functions f_j on Δ_j $(j=1, 2)$, a couple (f_1, f_2) is $\boldsymbol{H}$-resolutive.*

By Lemma 13 we have

Theorem 6. *A couple (X_1^*, X_2^*) of compactifications of X is $\boldsymbol{H}$-resolutive compactification if and only if for any bounded continuous function f_j on X_j^* $(j=1, 2)$, (f_1, f_2) is in $\boldsymbol{W}(X, \boldsymbol{H})$.*

Further we shall show the following

Theorem 7. *Suppose that the constant function 1 is in $\boldsymbol{W}^{(2)}(X)$. Then a couple (X_1^*, X_2^*) of two compactifications of X is $\boldsymbol{H}$-resolutive if and only if X_j^* is $\boldsymbol{H}_j$-resolutive compactification of the associated Brelot's harmonic space $(X, \boldsymbol{H}_j)$ $(j=1, 2)$. In this case we have*

$$H_1(f_1, f_2)=H_{f_1}^{(1)}+\int G_1(\cdot, y)H_{f_2}^{(2)}(y)d\alpha(y),\ H_2(f_1, f_2)=H_{f_2}^{(2)}$$

for any couple (f_1, f_2) of bounded continuous functions f_j on Δ_j $(j=1, 2)$.

Proof. Let f be any bounded continuous function on Δ_1(resp. Δ_2). Then $(f, 0)$ (resp. $(0, f)$) being $\boldsymbol{H}$-resolutive, f is $\boldsymbol{H}_1$-resolutive (resp. $\boldsymbol{H}_2$-resolutive) and $H_f^{(1)} =H_1(f, 0)$, $H_f^{(2)}=H_2(0, f)$. Conversely let (f_1, f_2) be any couple of bounded continuous functions f_j on Δ_j $(j=1, 2)$ and f_j^* be a bounded continuous function on X_j^* such that $f_j^*=f_j$ on Δ_j $(j=1, 2)$. Then $f_j^*\in \boldsymbol{W}^{(j)}(X)$ $(j=1, 2)$ by Theorem 4.4 in [6]. Since $\boldsymbol{W}^{(1)}(X)\times \boldsymbol{W}^{(2)}(X)=\boldsymbol{W}(X, \boldsymbol{H})$, we know that $(f_1^*, f_2^*)\in \boldsymbol{W}(X, \boldsymbol{H})$. Hence (f_1, f_2) is $\boldsymbol{H}$-resolutive by Lemma 13 and so (X_1^*, X_2^*) is $\boldsymbol{H}$-resolutive. The integral representation is shown by Lemma 12.

By Lemma 1 and Theorem 4.7 in [6], we have

Corollary. *Suppose that the constant function 1 is in $\boldsymbol{W}^{(1)}(X)$. If a couple (X_1^*, X_2^*) of two compactifications of X is $\boldsymbol{H}$-resolutive, then* $\mathrm{Supp}(\mu_x^*)=\Gamma_1$ *and* $\mathrm{Supp}(\nu_x^*)=\mathrm{Supp}(\lambda_x^*)=\Gamma_2$.

Definition 4. *Let (X_1^*, X_2^*) be a $\boldsymbol{H}$-resolutive compactification. A couple (a_1, a_2) of points $a_j\in\Delta_j (j=1, 2)$ is called $\boldsymbol{H}$-regular, if for any couple (f_1, f_2) of bounded continuous functions f_j on Δ_j $(j=1, 2)$,*

$$\lim_{x\to a_j} H_j(f_1, f_2)(x)=f_j(a_j)\ (j=1, 2).$$

If the constant function 1 is in $\boldsymbol{W}^{(2)}(X)$, then a couple (a_1, a_2) of points $a_j\in\Delta_j$ $(j=1, 2)$ is $\boldsymbol{H}$-regular if and only if a_j is $\boldsymbol{H}_j$-regular $(j=1, 2)$. Hence all couples (a_1, a_2) $\notin\Gamma$ are not $\boldsymbol{H}$-regular by Lemma 10.

Suppose that the constant function 1 is in $\boldsymbol{W}^{(1)}(X)$ and $\boldsymbol{W}^{(2)}(X)$. Let X_j^* be the Wiener compactification of $(X, \boldsymbol{H}_j)$ $(j=1, 2)$. Then X_j^* is $\boldsymbol{H}_j$-resolutive and all points in Γ_j are $\boldsymbol{H}_j$-regular ($j=1, 2$) by Corollary 5.1 and Theorem 5.4 in [6]. Hence in this case we have the following

Corollary. (c. f. Theorem 7 in [12]) *If the constant function 1 is in* $W^{(1)}(X)$ *and* $W^{(2)}(X)$, *then the Riquier's boundary value problem on the ideal boundaries has a unique solution (i. e. for any couple* (f_1, f_2) *of bounded continuous functions* f_j *on* Δ_j $(j=1, 2)$, *there exists a unique* $(u_1, u_2)\in \boldsymbol{H}(X)$ *such that* $\lim_{x\to a} u_j(x)=f_j(a)$ *for any* $a\in\Gamma_j(j=1, 2))$.

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